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Cranks and t-cores

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Summary. New statistics on partitions (called *cranks*) are defined which combinatorially prove Ramanujan's congruences for the partition function modulo 5, 7, 11, and 25. Explicit bijections are given for the equinumerous crank classes. The cranks are closely related to the t-core of a partition. Using q-series, some explicit formulas are given for the number of partitions which are t-cores. Some related questions for self-conjugate and distinct partitions are discussed.

1. Introduction

Let $p(n)$ be the number of partitions of n [1]. Dyson [7] proposed a combinatorial proof of Ramanujan's congruence

$$p(5n+4) \equiv 0 \mod 5 .$$

He defined an integral statistic on partitions, called the rank, whose value mod 5 split the set of partitions of $5n+4$ into 5 equal classes. He further conjectured that the rank also proved

$$p(7n+5) \equiv 0 \mod 7 ,$$

and hypothesized a statistic, called the crank, which would similarly prove

$$p(11n+6) \equiv 0 \mod 11 .$$

Atkin and Swinnerton-Dyer [5] proved Dyson's conjecture for 5 and 7. In [9], a crank for 11 as well as new cranks for 5 and 7 were found relative to vector partitions. Later, Andrews and Garvan [3] were able to define a crank, in terms of ordinary partitions, for 5, 7 and 11, thus completing the solution of Dyson's crank conjecture. New relations for the crank of partitions mod 8, 9 and 10 have been found in [10].

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Explicit bijections between these equinumerous classes have not been found in any of these three cases. This should be related to another possible approach to these problems: split the set of partitions of $5n + 4$ into $p(5n.+ 4)/5$ classes, each of size 5. Hopefully, the bijections among the five classes should be realized by a 5-cycle which acts on all partitions of $5n + 4$ with no fixed points. The orbits of the 5-cycle are the $p(5n + 4)/5$ classes. A crank statistic should be defined which is $\{0, 1, 2, 3, 4\}$ on an orbit. A 5-cycle, but no crank, was found by computer in [11]. In this paper, we complete this program by finding dihedral groups of size 10, 14, and 22 which act on the partitions of $5n + 4$, $7n + 5$, and $11n + 6$, respectively. The 5 (7 or 11) cycles in these groups have no fixed points, thus proving the congruences. We also define new statistics on partitions, i.e. new cranks, which split the set of partitions into equinumerous classes. A cycle in the dihedral groups increases the crank by 1, thus supplying an explicit bijection between the crank classes. Our definitions of the cycles and the cranks are completely combinatorial.

The key ingredients to our proof are two bijections for partitions and t-cores, which are given in §2. They allow us to find dihedral groups as symmetry groups of quadratic forms. There are interesting q-series attached to these forms, and these q-series imply some remarkable explicit formulas (Theorems 4 and 5 in §5) for the number of partitions which are t-cores, $t = 5$ or 7. The explicit cranks are given in Theorems 2 and 3, and in Proposition 1 of §3 and §4. A crank for $25n + 24$ is given in §6. Similar bijections and generating functions for self-conjugate partitions and partitions with distinct parts are given in §7 and §8.

We shall use the notation for generating functions from q-series, e.g.

$$(a;q)_k = \prod_{i=0}^{k-1} (1 - aq^i),$$

so

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

2. Two bijections

In this section we give two bijections relating partitions and t-cores.

First we set the notation. Let P be the set of all partitions. For any $\lambda \in P$, let $|\lambda|$ denote the number that λ partitions. Fix a positive integer t. Let P_{t-core} be the set of partitions which are t-cores. Recall that there are two equivalent definitions of a t-core: a partition λ is a t-core if, and only if, λ has no hook numbers that are multiples of t [13, 2.7.40]; or if, and only if, λ has no rim hooks that are multiples of t [13, p. 75]. We let $a_t(n)$ be the number of partitions of n which are t-cores.

The first bijection is well-known [13, 2.7.17].

Bijection 1. *There is a bijection* $\phi_1 : P \to P_{t-core} \times P \times P \times \ldots \times P$,

$$\phi_1(\lambda) = (\tilde{\lambda}, \lambda_0, \lambda_1, \ldots, \lambda_{t-1}),$$

such that $|\lambda| = |\tilde{\lambda}| + t\sum_{i=0}^{t-1} |\lambda_i|$.

The generating function identity equivalent to Bijection 1 is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^t;q^t)_\infty^t} \sum_{n=0}^{\infty} a_t(n)q^n . \tag{2.1}$$

The second bijection is on t-cores only.

Bijection 2. *There is a bijection* ϕ_2: $P_{t-core} \to \{\vec{n} = (n_0, n_1, \ldots, n_{t-1})$: $n_i \in \mathbb{Z}$, $n_0 + \ldots + n_{t-1} = 0\}$, *where*

$$|\tilde{\lambda}| = t\|\vec{n}\|^2/2 + \vec{b}\cdot\vec{n}, \quad \vec{b} = (0, 1, \ldots, t-1) .$$

The generating function identity equivalent to Bijection 2 is

$$\sum_{n=0}^{\infty} a_t(n)q^n = \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}} . \tag{2.2}$$

Clearly (2.1) and (2.2) imply

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}} . \tag{2.3}$$

We now give Bijection 2, and sketch Bijection 1.

Let $\tilde{\lambda}$ be a t-core. Define the vector $\vec{n} = \phi_2(\tilde{\lambda})$ in the following way. Label a cell in the ith row and jth column of $\tilde{\lambda}$ by $j - i \bmod t$. (This is called the t-residue diagram in [13, p. 84].) We also label the cells in column 0 in the same way, and call the resulting diagram the extended t-residue diagram. A cell is called exposed if it is at the end of a row. Region r of the extended t-residue diagram of $\tilde{\lambda}$ is the set of cells (i, j) satisfying $t(r-1) \leqq j - i < tr$. We now define n_i to be the maximum region of $\tilde{\lambda}$ which contains an exposed cell labeled i. Since column 0 contains infinitely many exposed cells, n_i is well-defined.

If an exposed cell labeled i lies in region r, then there is an exposed cell labeled i in each region $< r$. For example, in region $r-1$, if i is not exposed, then the rim hook whose northeast head is i in region r, and whose tail is $i+1$ in region $r-1$ has length t. (Note: If $i = t-1$, the rim hook will lie entirely in region r.) This contradicts the assumption that $\tilde{\lambda}$ is a t-core.

Note that the size of the Durfee square of $\tilde{\lambda}$ is the sum of the positive n_i's, since this is the number of exposed cells in positive regions.

We next verify that $n_0 + n_1 + \ldots + n_{t-1} = 0$. Let $\tilde{\lambda}'$ be the conjugate of $\tilde{\lambda}$. First we show that if $\phi_2(\tilde{\lambda}) = (n_0, n_1, \ldots, n_{t-1})$, then $\phi_2(\tilde{\lambda}') = (-n_{t-1}, -n_{t-2}, \cdots, -n_0)$. Let a cell labeled i be exposed in region r of $\tilde{\lambda}$, so that $i+1$ lies above i. In $\tilde{\lambda}'$, the cell corresponding to $i+1$ will be labeled $t-i-1$, lie in region $1-r$, and not be the last cell in its row. Thus $t-i-1$ is not exposed in region $1-r$ of $\tilde{\lambda}'$. This shows that if i is exposed in region r of $\tilde{\lambda}$, $r \leqq n_i$; then $t-1-i$ is not exposed in region r of $\tilde{\lambda}'$, $r \geqq 1 - n_i$. A similar argument shows that if a boundary cell labeled $t-i-1$ is not exposed in region $1-r$, then i is exposed in region r. This verifies the claim about $\phi_2(\tilde{\lambda}')$. Since the size of the Durfee square is unchanged by conjugation, the sum of the positive n_i's must be equal to minus the sum of the negative n_i's; that is, $n_0 + n_1 + \ldots + n_{t-1} = 0$.

The inverse map to ϕ_2 is easy to find: if $(n_0, n_1, \ldots, n_{t-1})$ is given, then the exposed cells in each region are known. This in turn gives the lengths of each row of the Ferrers diagram of $\tilde{\lambda}$, and thus $\tilde{\lambda}$.

Finally we show that if $\phi_2(\tilde{\lambda}) = \vec{n}$, then $\tilde{\lambda}$ is a partition of the exponent in Bijection 2. The number of cells to the right of the main diagonal of the Ferrers diagram of λ is

$$\sum_{n_i > 0} in_i + t\binom{n_i}{2}.$$

For the cells below the main diagonal, we take $\tilde{\lambda}'$, to find

$$-\sum_{n_j < 0} (t-1-j)n_j + t\binom{-n_j}{2}.$$

There are

$$\sum_{n_i > 0} n_i$$

cells on the main diagonal. The sum of these three terms is the exponent of q in (2.3).

We quickly sketch ϕ_1 using the notation in Bijection 2. Given λ, we construct t biinfinite words in the letters N and E, $w_0, \ldots, w_{t-1}$. The jth letter of w_i is E if i is exposed in region j of λ, otherwise the jth letter is N (not exposed). If λ is a t-core, then each word w_i is an infinite sequence of E's followed by an infinite sequence of N's. In this case, $\lambda = \tilde{\lambda}$, and $\lambda_i = \varnothing$ for all i. Otherwise there is an E to the right of some N. Find the rightmost E, say in position j. Find the rightmost N to the left of this E, say in position $k < j$. Delete a rim hook in λ whose northeast head is the cell corresponding to E, and whose southwest tail is adjacent to the cell corresponding to N. Place a part of size $j - k$ in λ_i. The new partition has N's in w_i from position j on, thus the E's have been pushed to the left. This operation can be done in any order whatsoever, to finally obtain the t-core of λ, $\tilde{\lambda}$. The parts of each λ_i appear in increasing order.

3. Cranks

In this section we give the cranks (Theorem 2) and the cycles which give bijections for these crank classes.

Let r be a residue class of t, and consider $p(tn + r)$. On the right side of (2.3), all exponents of q which are equivalent to $r \bmod t$ lie in the multisum. Since $\vec{n} \cdot \vec{1} = 0$, we have $\frac{1}{2}\|\vec{n}\|^2 \equiv 0 \bmod t$. Thus we have

$$\sum_{n=0}^{\infty} p(tn + r)q^{tn+r} = \frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{b} \equiv r \bmod t \\ \vec{n}\cdot\vec{1} = 0 \\ \vec{n} \in Z^t}} q^{\frac{1}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}}. \tag{3.1}$$

The symmetry groups of the quadratic forms in (3.1) have been computed by computer for $t \leqq 8$ in [11]. Until now, we have assumed that the class t and residue r are arbitrary. We shall see that the choices of $5n + 4$, $7n + 5$, and $11n + 6$ simplify (3.1).

For $5n + 4$ we now change variables so that the symmetry group is obviously the dihedral group D_5 of order 10. The five vectors $\vec{n}$ for $\phi_2(p(4))$ are

(1) $\vec{v}_0 = (1, -1, 0, 0, 0)$,
(2) $\vec{v}_1 = (0, 1, -1, 0, 0)$,
(3) $\vec{v}_2 = (0, 0, 1, -1, 0)$,
(4) $\vec{v}_3 = (0, 0, 0, 1, -1)$, and
(5) $\vec{v}_4 = (1, 1, 0, -1, -1)$.

They form a non-planar pentagon whose center $\vec{c} = (2/5, 1/5, 0, -1/5, -2/5)$.

We now change variables: write $\vec{n} = \alpha_0\vec{v}_0 + \ldots + \alpha_4\vec{v}_4$. A vector $\vec{n} \in \mathbb{Z}^5$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 4 \bmod 5$ if, and only if, $\vec{n} = \alpha_0 v_0 + \ldots + \alpha_4 v_4$ for some $\vec{\alpha} \in \mathbb{Z}^5$ such that $\vec{\alpha} \cdot \vec{1} = 1$.

It is easy to see that the quadratic form in (3.1) in terms of the vector $\vec{\alpha}$ is

$$5\|\vec{\alpha}\|^2 - 5(\alpha_0\alpha_1 + \alpha_1\alpha_2 + \ldots + \alpha_4\alpha_0) - 1 .$$

Thus we obtain

$$\sum_{n=0}^{\infty} a_5(5n + 4)q^{n+1} = \sum_{\substack{\vec{\alpha} \cdot \vec{1} = 1 \\ \vec{\alpha} \in \mathbb{Z}^t}} q^{Q(\vec{\alpha})} ,$$

where $Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \ldots + \alpha_4\alpha_0)$. Clearly the dihedral group D_5 is the automorphism group.

It is also clear that the 5-cycle which cyclically permutes the α_i's has no fixed points. Any fixed point $\vec{\alpha}$ must have all entries equal. Since $\vec{\alpha} \cdot \vec{1} = 1$, this would imply that the entries are not integral.

The construction of the quadratic form and the group are completely analogous for $7n + 5$ and $11n + 6$. We collect these results in a theorem.

Theorem 1. *For* $(t, r) = (5, 4), (7, 5)$ *or* $(11, 6)$,

$$\sum_{n=0}^{\infty} p(tn + r)q^{n+1} = \frac{1}{(q;q)_\infty^t} \sum_{\substack{\vec{\alpha} \cdot \vec{1} = 1 \\ \vec{\alpha} \in \mathbb{Z}^t}} q^{Q(\vec{\alpha})} , \tag{3.2}$$

where $Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - \sum_{i=0}^{t-1} \alpha_i\alpha_{i+1}$.

It should be noted that other identities for the generating functions of $p(tn + r)$, which yield Ramanujan's congruences, have been found by others. The most famous of these is Ramanujan's [19, (17) p. 213] identity for $p(5n + 4)$, which was considered by Hardy in agreement with MacMahon as Ramanujan's most beautiful identity [19, p. xxxv]. Ramanujan [19, (18) p. 213] also found an analog for $p(7n + 5)$. In fact for $t = 5^\alpha, 7^\alpha$ similar identities have been found by Watson [20]. Using the theory of modular functions an analog for $p(11n + 6)$ was found by Fine [8, (3.25)] and extended to higher powers of 11 by Atkin [4]. However, these other identities are of a different nature than (3.2) and do not yield obvious cranks. Further, Fine's identity for $t = 11$ and Ramanujan's identity for $t = 5$ or $t = 7$ are dissimilar so it is surprising that an identity like (3.2) holds for all three values $t = 5$, 7 and 11.

Finally we come to the cranks. An obvious statistic, which is increased by 1 mod 5 each time the α's are cyclically permuted is $\sum_{i=0}^{4} i\alpha_i$. So in terms of the $\vec{\alpha}$ basis the crank is clear, but in terms of the $\vec{n}$ basis (given combinatorially in §2) the crank is not clear.

Theorem 2. *A crank statistic for partitions λ of $5n+4$, $7n+5$, or $11n+6$ is given by the following algorithm.*

(1) *Find the t-core $\tilde{\lambda}$ of λ, ($t = 5$, 7, or 11) by Bijection* 1.
(2) *Find $\phi_2(\tilde{\lambda}) = \vec{n}$ by Bijection* 2.
(3) *Let crank(λ) be the following* mod t *linear combination:*

($t = 5$) $4n_0 + n_1 + n_3 + 4n_4$ *for* $5n+4$,
($t = 7$) $4n_0 + 2n_1 + n_2 + n_4 + 2n_5 + 4n_6$ *for* $7n+5$,
($t = 11$) $4n_0 + 9n_1 + 5n_2 + 3n_3 + n_4 + n_6 + 3n_7 + 5n_8 + 9n_9 + 4n_{10}$ *for* $11n+6$.

The new crank is not equal to the rank or the old crank. One can verify this on the 5 partitions of 9 which are 5-cores: 621, 522, 51111, 33111, and 321111.

4. More cranks

The cranks in §3 depend only upon the t-core $\tilde{\lambda}$ of λ, and thus use Bijections 1 and 2. In this section we give two simpler ways (Proposition 1 and Theorem 3) of computing crank(λ), (one which is a polynomial in the parts of λ), thus avoiding Bijections 1 and 2.

First, let $r_k(\lambda)$ be the number of cells in the t-residue diagram of λ which are labeled k mod t. Suppose that λ is a t-core. A relation between the r_k's and the n_i's can be found by counting the cells in a row ending with i in region n_i. It is

$$r_k = \sum_{i=0}^{t-1} (n_i^2/2 + n_i \chi(i \geqq k)) \tag{4.1}$$

where

$$\chi(S) = \begin{cases} 1/2 & \text{if } S \text{ is true,} \\ -1/2 & \text{if } S \text{ is false .} \end{cases}$$

It is easy to see that (4.1) implies

$$n_k = r_k - r_{k+1}\,, \tag{4.2}$$

so that the relation in (4.1) for r_0 for t-cores becomes

$$r_0 = \sum_{i=0}^{t-1} (r_i^2 - r_i r_{i+1})\,.$$

The cranks in Theorem 2 are defined by a linear combination of the n_i's. Using (4.2), we can find mod t equivalent linear combinations of the r_k's.

Proposition 1. *The following* mod t *linear combinations of the t-residue numbers r_k are crank statistics*:

(1) $r_1 + 2r_2 - 2r_3 - r_4$ *for* $5n + 4$,
(2) $2r_1 + r_2 + r_3 - r_4 - r_5 - 2r_6$ *for* $7n + 5$,
(3) $-6r_1 - 4r_2 - 2r_3 - 2r_4 - r_5 + r_6 + 2r_7 + 2r_8 + 4r_9 + 6r_{10}$ *for* $11n + 6$.

Proof. We already know from Theorem 2 that certain linear combinations give the crank for t-cores, e.g. $(2, -1, 1, -2)$ for $t = 5$. If we multiply this linear combination by 3, we obtain (1) for t-cores. We reverse the sign to obtain (2), and (3) is immediate from Theorem 2. Now consider (1)–(3) as definitions of a crank for any partition λ, not just t-cores. Removing a rim hook of length t from λ reduces each r_k by one. The above linear combinations do not change, since the sum of all of the coefficients is zero. Thus the crank can be defined by Proposition 1 for all partitions λ. □

Next, we give a definition of a crank based upon the parts of λ: $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m$.

Theorem 3. *For $t = 5, 7$ or 11, let $p_t(x) = (x - (t-1)/2)^{t-3}$ mod t. A crank statistic for partitions λ: $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m$ of $tn + r$ (where $r = 4, 5$, and 6, respectively) is given by*

$$crank(\lambda) = \sum_{i=1}^{m} (p_t(\lambda_i - i) - p_t(i-1)) \mod t .$$

Proof. Let a_i be the coefficient of r_i in Proposition 1. Recall that $a_1 + a_2 + \ldots + a_{t-1} = 0$.

First we evaluate the contribution to the crank in Proposition 1 from the first row of λ. The t-residue diagram of λ has an $x \equiv \lambda_1 - 1 \bmod t$ at the end of the first row. Then the contribution of this row to the crank is

$$a_1 + a_2 + \ldots + a_x . \tag{4.3}$$

It remains to find a polynomial whose values at $x = 0, 1, \ldots, t-1$ agree with (4.3) mod t. Here are such polynomials:

(1) $q_5(x) = 3(p_5(x) - 4)$ for $t = 5$,
(2) $q_7(x) = -(p_7(x) - 4)$ for $t = 7$, and
(3) $q_{11}(x) = p_{11}(x) - 4$ for $t = 11$.

Thus we have shown that the first row contributes $q_t(x) \equiv q_t(\lambda_1 - 1) - q_t(0) \bmod t$.

Consider row i of the t-residue diagram of λ: the first entry is $1 - i \bmod t$, and the final entry is $\lambda_i - i \bmod t$. The contribution of this row to the crank is

$$a_1 + a_2 + \ldots + a_{\lambda_i - i} + (a_{t+1-i} + \ldots + a_{t-1}) \equiv$$

$$a_1 + a_2 + \ldots + a_{\lambda_i - i} - (a_1 + \ldots + a_{t-i}) \equiv q_t(\lambda_i - i) - q_t(t-i) \bmod t.$$

Thus

$$crank(\lambda) = \sum_{i=1}^{m} (q_t(\lambda_i - i) - q_t(t-i)) \bmod t .$$

Since $q_t(t-1-i) \equiv q_t(i) \bmod t$, and $q_t(i)$ is a linear function of $p_t(i)$, the result follows. □

5. Associated q-series

From (2.1) there is an explicit generating function for the partitions of n which are t-cores. We investigate these functions for $t = 5$, 7, and 11, and give explicit formulas for $a_5(n)$ and $a_7(n)$ in Theorems 4 and 5.

The five cycle in §3 implies that $a_5(5n+4) \equiv 0 \bmod 5$. However, much more is true,

$$a_5(5n+4) = 5a_5(n)\,. \tag{5.1}$$

We give two proofs of (5.1): one which is analytic, and proves more (Theorem 4), and another which is a bijection.

Theorem 4. *Let* $n+1 = 5^c p_1^{a_1} \ldots p_s^{a_s} q_1^{b_1} \ldots q_t^{b_t}$ *be the prime factorization of* $n+1$ *into primes* $p_i \equiv 1, 4 \bmod 5$, *and* $q_j \equiv 2, 3 \bmod 5$. *Then*

$$a_5(n) = 5^c \prod_{i=1}^{s} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{j=1}^{t} \frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}\,.$$

Proof. Clearly (2.1) implies

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}\,.$$

However an identity of Ramanujan ([2, (3.46)] or [6]) is

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=0}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}\,.$$

where $\left(\frac{a}{b}\right)$ is the Legendre symbol. Clearly the right side of (5.2) is

$$\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{d}{5}\right)\frac{n}{d}q^n\,.$$

so

$$a_5(n-1) = \sum_{d|n}\left(\frac{d}{5}\right)\frac{n}{d}\,. \tag{5.3}$$

Since $(\frac{d}{5})/d$ is multiplicative, (5.3) and [12, Theorem 265] implies that $a_5(n-1)$ is multiplicative. It remains to evaluate $a_5(p^a-1)$ for primes p, which can be done by (5.3). □

The combinatorial proof of (5.1) is explicit: given a partition $\tilde{\lambda}$ of n which is a 5-core, find five partitions $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$, of $5n+4$ which are also 5-cores. Let $\phi_2(\tilde{\lambda}) = \vec{m}$. We now give the image under ϕ_2 of the five partitions of $5n+4$.

(1) $\phi_2(\theta_0) = (m_1 + 2m_2 + 2m_4 + 1,\ -m_1 - m_2 + m_3 + m_4 + 1,\ 2m_1 + m_2$
$+ 2m_3,\ -2m_2 - 2m_3 - m_4 - 1,\ -2m_1 - m_3 - 2m_4 - 1)$,

$$(2)\ \phi_2(\theta_1) = (2m_1 + 2m_4 + 1 + m_3,\ -1 - m_2 - 2m_3 - 2m_4,$$
$$-2m_1 - 2m_2 - m_4,\ m_1 + 2m_2 + 2m_3,\ -m_1 + m_2 + m_4 - m_3),$$

$$(3)\ \phi_2(\theta_2) = (m_1 - m_2 - m_4 + m_3,\ 2m_2 + m_4 + 1 + 2m_3,\ -m_1 - 2m_4$$
$$-1 - 2m_3,\ -2m_1 - 2m_2 - m_3,\ 2m_1 + m_2 + 2m_4),$$

$$(4)\ \phi_2(\theta_3) = (-2m_1 - m_2 - 2m_4,\ -m_1 - 2m_2 - 2m_3,\ 2m_2 + 2m_4 + 1 + m_3,$$
$$m_1 + m_2 - m_4 - 1 - m_3,\ 2m_1 + 2m_3 + m_4),$$

$$(5)\ \phi_2(\theta_4) = (-2m_1 - m_4 - 2m_3,\ 2m_1 + 2m_2 + m_3,\ m_1 - m_2 + m_4 - m_3,$$
$$m_2 + 2m_4 + 1 + 2m_3,\ -m_1 - 2m_2 - 2m_4 - 1).$$

We delete the verification of this claim. In fact, these five partitions θ_i form an orbit under the five cycle, and

$$crank(\theta_i) \equiv 3i \mod 5\,.$$

We denote by γ_n the $5 -$ to -1 map from the set of 5-cores of $5n + 4$ to the set of 5-cores of n, $\gamma_n(\theta_i) = \tilde{\lambda}$ for $0 \leqq i \leqq 4$.

For $t = 7$, we have $a_7(7n + 5) \equiv 0 \bmod 7$. It is not true that $a_7(7n + 5) = 7a_7(n)$. There are however two multiplicative functions here. The identity which is analogous to (5.2) is due to Ramanujan [8, p. 159]

$$8q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} + q(q^7;q^7)_\infty^3 (q;q)_\infty^3 = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n(1+q^n)}{(1-q^n)^3}\,. \tag{5.4}$$

Let $a'(n)$, $b(n)$, $c(n)$ denote the coefficient of q^n in the expansion of each of the three functions in (5.4) so that

$$8a_7(n) = a'(n+2)\,,$$
$$a'(n) + b(n) = c(n)\,.$$

We will give in Lemmas 1 and 2 explicit formulas for $b(n)$ and $c(n)$, resulting in an explicit formula for $a_7(n)$ in Theorem 5.

The proof of Theorem 4 analogously establishes the multiplicativity of $c(n)$. We delete the details. The resulting explicit formula is given in Lemma 1.

Lemma 1. *If* $n = 7^c p_1^{a_1} \ldots p_s^{a_s} q_1^{b_1} \ldots q_t^{b_t}$ *is the prime factorization of* n *into primes* $p_i \equiv 1, 2, 4 \bmod 7$, *and* $q_j \equiv 3, 5, 6 \bmod 7$, *then*

$$c(n) = 7^{2c} \prod_{i=1}^{s} \frac{p_i^{2a_i+2} - 1}{p_i^2 - 1} \prod_{j=1}^{t} \frac{q_i^{2b_j+2} + (-1)^{b_j}}{q_j^2 + 1}\,.$$

To establish the multiplicativity of the sequence $b(n)$, we apply the theory of modular forms.

Proposition 2. *The sequence* $b(n)$ *is multiplicative, moreover*

$$b(p^m) = b(p)b(p^{m-1}) - \left(\frac{p}{7}\right) p^2 b(p^{m-2})$$

for any prime p *and integer* $m \geqq 2$.

Proof. If $q = e^{2\pi iz}$, the generating function $B(z)$ for $b(n)$ (see (5.4)) can be considered a function of z in the upper half plane. In terms of the classical η-function it is $B(z) = (\eta(z)\eta(7z))^3$, which is a cusp form of level 7, weight 3, and character $\chi(n) = (\frac{n}{7})$ [14, Prob. 12, 13, p. 145]. Since this space of cusp forms is one-dimensional, $B(z)$ must be an eigenform for the Hecke operators T_n. This proves the multiplicativity of $b(n)$, and the three term recurrence follows from the Euler product for the associated Dirichlet series [14, p. 160 (5.11)]. □

To find an explicit formula for $b(n)$, we need only find $b(p)$ for all primes p. The three term recurrence in Proposition 2 gives $b(p^m)$, and multiplicativity gives $b(n)$ for all n. The next proposition gives the values of $b(p)$.

Proposition 3. *Suppose p is an odd prime. If $p \equiv 3, 5$ or $6 \bmod 7$, then $b(p) = 0$. If $p \equiv 1, 2$ or $4 \bmod 7$, let $p = x^2 + 7y^2$, where x and y are positive integers. Then $b(p) = 2(x^2 - 7y^2)$.*

Remark. The result also holds for $p = 2$ except in this case $x = y = \frac{1}{2}$.

Proof. From Jacobi's identity for $(q;q)_\infty^3$ [12, Thm 357] we find that

$$b(p) = \sum_{\substack{m,n \geqq 0 \\ (2m+1)^2 + 7(2n+1)^2 = 8p}} (-1)^{m+n}(2m+1)(2n+1) .$$

We need to determine all of the solutions (m, n) to

$$8p = (2m+1)^2 + 7(2n+1)^2 , \tag{p1}$$

where $m, n \geqq 0$. Clearly the right side is a quadratic residue mod 7, so if $p \equiv 3, 5$ or 6 mod 7, there are no solutions (m, n) to (p1) and $b(p) = 0$.

Thus we can assume that $p \equiv 1, 2$ or 4 mod 7. We first show in this case that p has a unique representation $p = x^2 + 7y^2$ for positive integers x, y. We work in $K = \mathbb{Q}(\sqrt{-7})$, whose ring of algebraic integers, $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$, is a unique factorization domain [12, Th. 246]. Since

$$1 = \left(\frac{p}{7}\right) = \left(\frac{-7}{p}\right) \quad \text{(by quadratic reciprocity)} ,$$

p divides $a^2 + 7$ for some integer a. Since p does not divide $a \pm \sqrt{-7}$, p is not prime. Thus p splits

$$p = \beta\bar{\beta} ,$$

where $\beta = x + y\sqrt{-7}$ is prime in $\mathcal{O}_K$ and $x, y > 0$. Now, $x, y \in \mathbb{Z}$, otherwise $x = m + \frac{1}{2}$, $y = n + \frac{1}{2}$ (for some $m, n \in \mathbb{Z}$), $p = m(m+1) + 7n(n+1) + 2$ which is even and contradicts the assumption that p was odd. Hence $p = \beta\bar{\beta} = x^2 + 7y^2$ for some positive integers x, y. The uniqueness of the representation follows from unique factorization, and the only units being ± 1.

We next show that (p1) always has two solutions (m, n), whose contribution to $b(p)$ is $2(x^2 - 7y^2)$. (p1) is equivalent to $2p = \gamma\bar{\gamma}$, where

$$\gamma = \frac{(2m+1) + (2n+1)\sqrt{-7}}{2} = \frac{x_1 + y_1\sqrt{-7}}{2}, \; x_1, y_1 > 0 .$$

Let

$$2 = \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right) = \pi\bar{\pi}\,.$$

It follows that $2p$ factors as

$$2p = (\pi\beta)(\bar{\pi}\bar{\beta}) = (\pi\bar{\beta})(\bar{\pi}\beta) = \gamma\bar{\gamma}\,.$$

By unique factorization we must have $\gamma = \pm\pi\beta$, $\pm\bar{\pi}\beta$, $\pm\bar{\pi}\bar{\beta}$, or $\gamma = \pm\pi\bar{\beta}$. This allows eight possible solutions (x_1, y_1), six of which violate $x_1, y_1 > 0$. The two solutions can be explicitly given, depending upon which of the three inequalities $0 = y < 7y < x$, $0 < x < y < 7y$, or $0 < y < x < 7y$ occur. In each of these three cases we find $b(p) = 2(x^2 - 7y^2)$ as required. □

For $p \equiv 1$, 2 or 4 mod 7 note that Proposition 3 implies that $b(p) \neq 0$, since $b(p) = 0$ would imply that $x^2 = 7y^2$, and thus $p = 14y^2 \equiv 0 \bmod 7$. We also note that $b(p) = 2(x^2 - 7y^2) = \beta^2 + \bar{\beta}^2$ where $p = \beta\bar{\beta}$, $\beta = x + y\sqrt{-7}$ and $x, y > 0$. Using the three term recurrence in Proposition 2 and the value of $b(p)$ in Proposition 3, the value of $b(p^m)$ is easily found. It is given in the next lemma.

Lemma 2. *Let $n = 7^c p_1^{a_1} \ldots p_s^{a_s} q_1^{b_1} \ldots q_t^{b_t}$ be the prime factorization of n into primes $p_i \equiv 1, 2, 4 \bmod 7$, and $q_j \equiv 3, 5, 6 \bmod 7$. Let $p_i = x_i^2 + 7y_i^2$ for some positive integers x_i, y_i in the case when p_i is odd and $x_1 = y_1 = \frac{1}{2}$ for the case $p_1 = 2$. Then*

$$b(n) = 0 \text{ if some } b_j \text{ is odd}\,,$$

$$b(n) = (-7)^c \prod_{i=1}^{s} \frac{\beta_i^{2a_i+2} - \bar{\beta}_i^{2a_i+2}}{\beta_i^2 - \bar{\beta}_i^2} \prod_{j=1}^{t} q_j^{b_j} \text{ otherwise}\,,$$

where $\beta_i = x_i + y_i\sqrt{-7}$.

Note that Lemma 2 implies that $b(p^m) = 0$ if, and only if, $p \equiv 3$, 5 or 6 mod 7 and m is odd. We can now state the explicit formula for $a_7(n)$.

Theorem 5. *Let $n + 2 = 7^c p_1^{a_1} \ldots p_s^{a_s} q_1^{b_1} \ldots q_t^{b_t}$ be the prime factorization of $n + 2$ into primes $p_i \equiv 1, 2, 4 \bmod 7$, and $q_j \equiv 3, 5, 6 \bmod 7$, then*

$$a_7(n) = \tfrac{1}{8}(c(n+2) - b(n+2))\,.$$

where $c(n)$ and $b(n)$ are given in Lemmas 1 and 2 respectively.

We state relations which are corollaries of Theorems 4 and 5.

Corollary 1. *The following relations hold:*

(1) $a_5(5^\alpha m - 1) = 5^\alpha a_5(m-1) \equiv 0 \bmod 5^\alpha$,
(2) $a_7(7^\alpha m - 2) \equiv 0 \bmod 7^\alpha$,
(3) $a_7(49n + 19) = 49a_7(7n + 1)$,
(4) $a_7(49n + 33) = 49a_7(7n + 3)$,
(5) $a_7(49n + 40) = 49a_7(7n + 4)$.

Proof. (1)–(5) are immediate from Theorems 4 and 5. □

For $a_{11}(n)$ there are identities for cusp forms which prove

$$a_{11}(11^a m - 5) \equiv 0 \bmod 11^a \tag{5.5}$$

and

$$a_{11}(43n - 5) \equiv 0 \bmod 44 \text{ for } (n, 43) = 1 . \tag{5.6}$$

We delete the proofs of (5.5) and (5.6).

6. A crank for $p(25n + 24)$

We can define a group G which acts on the set of partitions of $tn + r$: $G = G_1 \times S_t$, where G_1 is the automorphism group of the quadratic form in Bijection 2, and S_t is the symmetric group on t letters. For example, for $5n + 4$, $G = D_5 \times S_5$, where D_5 is the dihedral group of order 10. We have concentrated on the action of G_1 for congruences. We can also use S_t for congruences, and do so in this section to find a crank for

$$p(25n + 24) \equiv 0 \bmod 25 . \tag{6.1}$$

The group $G = D_5 \times S_5$ for $5n + 4$ contains two commuting five cycles which generate a subgroup of order 25. If each five cycle has no fixed points, then G will have no fixed points, and the orbits of G will have size 25. We have seen that the five cycle inside D_5 has no fixed points. Since the five cycle in S_5 could have fixed points, we consider two cases to define the crank, which will be an ordered pair of integers, each between 0 and 4. This ordered pair could be considered as the base 5 expansion for a number between 0 and 24.

Theorem 6. *A crank statistic for partitions λ of $25n + 24$ is given by the following algorithm.*

(1) *Find $\phi_1(\lambda) = (\tilde{\lambda}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ given by Bijection 1.*

(2) *Let crank(λ) be the following ordered pair of integers:*

Suppose that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, so that $\tilde{\lambda}$ is a partition of $25s + 24$ for some s. Let $\mu_1 = \gamma_{5s+4}(\tilde{\lambda})$ and $\mu_2 = \gamma_s(\mu_1)$ be partitions of $5s + 4$ and s respectively (see §5). Put

$$crank(\lambda) = (crank(\mu_1), crank(\mu_2)) . \tag{i}$$

Suppose that some $\lambda_i \neq \lambda_j$, so that the five cyclic permutations of $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are distinct. Order these five tuples in lex order, and define the crank of the ith tuple to be $i - 1$. Put

$$crank(\lambda) = (crank(\tilde{\lambda}), crank((\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4))) . \tag{ii}$$

Proof. The vectors $\gamma_n(\theta_i) = \tilde{\lambda}$ in §5 form an orbit under $C_5 \leqq G_1$, so that the cranks of these five partitions are $\{0, 1, 2, 3, 4\}$.

The second part of the definition similarly interprets $C_5 \times C_5$, as a subgroup of G. □

In Theorem 6(2)(ii) the lex order crank can be replaced by the following statistic. Find $\phi_1(\lambda)$, and if

$$|\lambda_0| + |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| \equiv 0 \bmod 5 ,$$

define the crank as in Theorem 6(2)(i). Otherwise define

$$crank(\lambda) = (crank(\tilde{\lambda}), |\lambda_1| + 2|\lambda_2| + 3|\lambda_3| + 4|\lambda_4| \bmod 5) .$$

It is clear that the five cyclic permutations of the λ_i will give five distinct cranks for the second component.

7. Self-conjugate partitions

In this section we consider the restriction of Bijections 1 and 2 to self-conjugate partitions. Let $asc_t(n)$ be the number of t-cores which are self-conjugate.

In Bijection 1, suppose that λ is self-conjugate. Then $\tilde{\lambda}$ must be self-conjugate, and $\lambda'_i = \lambda_{t-i-1}$, $0 \leqq i \leqq t-1$. Since $(-q;q^2)_\infty$ is the generating function for self-conjugate partitions, the self-conjugate version of (2.1) is

$$(-q;q^2)_\infty = \frac{1}{(q^{2t};q^{2t})_\infty^{t/2}} \sum_{n=0}^{\infty} asc_t(n) q^n \quad \text{for } t \text{ even} , \tag{7.1a}$$

and

$$(-q;q^2)_\infty = \frac{(-q^t;q^{2t})_\infty}{(q^{2t};q^{2t})_\infty^{(t-1)/2}} \sum_{n=0}^{\infty} asc_t(n) q^n \quad \text{for } t \text{ odd} . \tag{7.1b}$$

While there are congruences for $q_0(n)$, the number of self-conjugate partitions of n, e.g. $q_0(125n + 99) \equiv 0 \bmod 5$, it is not obvious how to prove these congruences from this approach. Nevertheless, for $t = 5$ there is a nice Lambert series, and a theorem analogous to Theorems 4 and 5.

The Lambert series is

$$q(-q;q^2)_\infty \frac{(q^{10};q^{10})_\infty^2}{(-q^5;q^{10})_\infty} = \sum_{n=1}^{\infty} \chi_{20}(n) \frac{q^n}{1-q^n} , \tag{7.2}$$

where

$$\chi_{20}(n) = \begin{cases} 0 & \text{if } (n, 10) > 1, \\ (-1)^{\frac{1}{2}(n-1)} & \text{otherwise} . \end{cases}$$

Thus only odd n occur in (7.2), and the appropriate signs for odd n mod 20 are $+ - 0 - + - + 0 + -$. (7.2) follows from the ${}_6\psi_6$ summation formula [2, (3.1)]. The identity (7.2) also follows from an identity of Ramanujan [21, (1) p. 60] and Jacobi's formula for $r_2(n)$, the number of representations of n as a sum of two squares [12, Thm 278]. The functions $r_2(n)$ and $asc_5(n)$ are related by

$$r_2(n) - r_2\left(\frac{n}{5}\right) = 4 asc_5(n-1) . \tag{7.3}$$

As before we obtain from (7.2) the following theorem.

Theorem 7. *Let* $n+1 = 2^c 5^d p_1^{a_1} \dots p_s^{a_s} q_1^{b_1} \dots q_t^{b_t}$ *be the prime factorization of* $n+1$ *into primes* $p_i \equiv 1 \bmod 4$, *and* $q_j = 3 \bmod 4$. *Then*

$$asc_5(n) = \prod_{i=1}^{s} (a_i + 1) \prod_{j=1}^{t} (1 + (-1)^{b_j})/2 .$$

A surprising corollary results.

Corollary 2. *The following relations hold*:

(1) $asc_5(2n+1) = asc_5(n)$,
(2) $asc_5(5n+4) = asc_5(n)$,
(3) $asc_5(n) = 0$ *if and only if there exists a prime* $q \equiv 3 \bmod 4$ *and an odd integer* b *such that* q^b *exactly divides* $n+1$.

There is a nice combinatorial proof of Corollary 2(2). Suppose $\tilde{\lambda}$ is a self-conjugate partition of $5n+4$ which is a 5-core. It is easy to see that only θ_0 of §4 is self-conjugate. Thus, the restriction of the map γ_n to self-conjugate partitions is a bijection.

A similar bijection exists for Corollary 2(1). Just replace (n_1, n_2) in (7.4) by $(-n_1 + n_2,\ -n_1 - n_2 - 1)$. We do not have a combinatorial proof of Corollary 2(3).

In Bijection 2, clearly we have $n_i = -n_{t-1-i}$, so the generating function becomes

$$\sum_{n=0}^{\infty} asc_t(n) q^n = \sum_{\vec{n} \in \mathbb{Z}^{[t/2]}} q^{t\|\vec{n}\|^2 + \vec{c}\cdot\vec{n}}. \tag{7.4}$$

where

$$\vec{c} = \begin{cases} (2, 4, \dots, t-1) & \text{for } t \text{ odd}, \\ (1, 3, \dots, t-1) & \text{for } t \text{ even}, \end{cases}$$

and $[t/2]$ is the integral part of $t/2$. Using (7.4) dissections of $(-q;q^2)_\infty$ due to Kolberg [16] can be easily derived.

8. Partitions with distinct parts

A bijection analogous to Bijection 1 for partitions with distinct parts has been given by Morris and Yaseen [17]. In this section we give the appropriate version of Bijections 1 and 2 for these partitions.

We begin with notation. Let $pd(n)$ be the number of partitions λ of n with distinct parts. Given such a λ, the shifted Ferrers diagram of λ, $S(\lambda)$, is the Ferrers diagram of λ with the ith row shifted to the right by $i-1$ cells. The doubled partition of λ, denoted $\lambda\lambda$, is defined by adding λ_i cells to the $i-1$st column of $S(\lambda)$. For example, if $\lambda = 31$, then $\lambda\lambda = 431$. We let DD denote the set of doubled distinct partitions $\lambda\lambda$. We define the t-core of $\lambda\lambda$ as the usual t-core of $\lambda\lambda$. Let DD_{t-core} be the set of partitions which are t-cores of doubled distinct partitions, and let $add_t(n)$

be the number of these which are partitions of n. We state without proof that $DD_{t-core} \subset DD$, thus $add_t(n) = 0$ if n is odd.

By restricting Bijection 1, we find a bijection for doubled distinct partitions. We must take the conjugate of the λ_0 in ϕ_1 to find λ_0 for ϕ_3.

Bijection 3. *There is a bijection* ϕ_3: $DD \to DD_{t-core} \times DD \times P \times \ldots \times P$,

$$\phi_1(\lambda\lambda) = (\lambda\tilde{\lambda}, \lambda_0, \lambda_1, \ldots, \lambda_{t-1}),$$

such that $2|\lambda| = |\lambda\lambda| = |\lambda\tilde{\lambda}| + t\sum_{i=0}^{t-1} |\lambda_i|$, *and* $\lambda_{t-i} = \lambda'_i$, *for* $1 \leqq i \leqq t-1$.

The generating function identities equivalent to Bijection 3 are

$$\sum_{n=0}^{\infty} pd(n)q^{2n} = \frac{(-q^{2t};q^{2t})_\infty}{(q^{2t};q^{2t})_\infty^{(t-1)/2}} \sum_{n=0}^{\infty} add_t(n)q^n \quad \text{for } t \text{ odd}. \tag{8.1a}$$

and

$$\sum_{n=0}^{\infty} pd(n)q^{2n} = \frac{(-q^{t};q^{t})_\infty}{(q^{2t};q^{2t})_\infty^{(t-2)/2}} \sum_{n=0}^{\infty} add_t(n)q^n \quad \text{for } t \text{ even}. \tag{8.1b}$$

Bijection 4. *There is a bijection* ϕ_4: $DD_{t-core} \to \{\vec{n} = (n_0, n_1, \ldots, n_{t-1})$: $n_i \in \mathbb{Z}$, $n_0 = 0$, $n_i = -n_{t-i}$, $1 \leqq i \leqq t-1\}$, *where*

$$|\lambda\tilde{\lambda}| = t\|\vec{n}\|^2/2 + \vec{b}\cdot\vec{n}, \quad \vec{b} = (0, 1, \ldots, t-1).$$

The generating function identity equivalent to Bijection 4 is

$$\sum_{n=0}^{\infty} add_t(n)q^n = \sum_{\vec{n} \in \mathbb{Z}^{[(t-1)/2]}} q^{t\|\vec{n}\|^2 + \vec{d}\cdot\vec{n}}. \tag{8.2}$$

where

$$\vec{d} = \begin{cases} (1, 3, \ldots, t-2) & \text{for } t \text{ odd}, \\ (2, 4, \ldots, t-2) & \text{for } t \text{ even}. \end{cases}$$

9. Remarks

Naturally it is desirable to prove the general congruence for powers of 5, 7, and 11 found in [4, 20]. It would be natural to assume that the symmetry group of the quadratic form for $t = 5^a7^b11^c$ contains a cycle of the appropriate length. We have not been able to find this cycle. Our crank for $p(25n + 24)$ was inspired by Bailey's ([6], [2, p. 465]) proof of Ramanujan's congruence for $p(5n + 4)$. In fact Bailey's method also yields the congruence for $p(25n + 24)$. It may be possible to extend our methods to obtain a crank for $p(49n + 47)$ and $p(121n + 116)$. First we need to find a crank for $a_7(49n + 47)$ and $a_{11}(121n + 116)$. Our present method does not succeed in these cases since there is no analog of (5.1). It may be easier to find a mod 49 crank for $p(49n + 19)$, $p(49n + 33)$, $p(49n + 40)$ in view of Corollary 1(3), (4), (5).

In [11] the symmetry groups of the forms associated with $p(tn + r)$ for $t \leqq 8$ and all r were computed. The symmetry groups for $p(11n + r)$ for $r \neq 6$ need to be

found. There is reason to believe these groups are larger than C_2. It seems that an unusually high proportion of the coefficients of $a_{11}(n)$ are divisible by 11 so there may be more 11-cycles. A higher proportion of the coefficients are divisible by 5. In fact, using modular forms which appear in an analog of (5.4) for $t = 11$, we can prove the following theorem.

Theorem 8. *If* $(\frac{n}{11}) = 1$ *and* n *not divisible by* 5, *then* $a_{11}(n - 5) = 0 \bmod 5$.

We can prove by a bijection from (7.4) that

$$asc_7(4n + 6) = asc_7(n) . \tag{9.1}$$

Further, by utilizing Gauss' result for sums of three squares [12, p. 311] it can be shown that

$$asc_7(n) = 0 \quad \text{if and only if } n + 2 = 4^a(8m + 1) . \tag{9.2}$$

(This observation was made by Doug McDoniel.)

Kolberg [15, 16] has given several infinite products for dissections of $p(n)$, $q(n)$ and $q_0(n)$. Many of these results (for $q(n)$ and $q_0(n)$) follow easily from (7.4), (8.2) and the Jacobi triple product formula.

John Stembridge informed the authors of the existence of [17, 18] for distinct partitions. It should be noted that the definition of t-core given in §8 for t-even does not agree with [17, 18].

References

1. Andrews, G.E.: The theory of partitions. Encyclopedia of Mathematics and Its Applications, Vol. 2. Rota, G.-C. (ed.), Reading, MA: Addison-Wesley 1976 (reissued by Cambridge Univ. Press, London and New York, 1985)
2. Andrews, G.E.: Applications of basic hypergeometric functions. SIAM Rev. **16**, 441–484 (1975)
3. Andrews, G.E., Garvan, F.G.: Dyson's crank of a partition. Bull. Am. Math. Soc. **18**, 167–171 (1988)
4. Atkin, A.O.L.: Proof of a conjecture of Ramanujan. Glasgow Math. J. **8**, 14–32 (1967)
5. Atkin, A.O.L., Swinnerton-Dyer, P.: Some properties of partitions. Proc. Lond. Math. Soc., III. Ser. **4**, 84–106 (1954)
6. Bailey, W.N.: A note on two of Ramanujan's formulae. Q. J. Math. Oxf. II. Ser. **3**, 29–31 (1952)
7. Dyson, F.: Some guesses in the theory of partitions. Eureka (Cambridge), **8**, 10–15 (1944)
8. Fine, N.: On a system of modular functions connected with the Ramanujan identities. Tohoku Math. J. **8**, 149–164 (1956)
9. Garvan, F.: New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and11. Trans. Am. Math. Soc. **305**, 47–77 (1988)
10. Garvan, F.: The crank of partitions mod 8, 9 and 10. Preprint
11. Garvan, F., Stanton, D.: Sieved partition functions and q-binomial coefficients. Math. Comput. (to appear)
12. Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers. London: Oxford Univ. Press 1979
13. James, G., Kerber, A.: The Representation Theory of the Symmetric Group. Reading, MA: Addison-Wesley 1981
14. Koblitz, N.: Introduction to Elliptic Curves and Modular Forms. New York: Springer, 1984

15. Kolberg, O.: Some identities involving the partition function. Math. Scand. **5**, 77–92 (1957)
16. Kolberg, O.: An elementary discussion of certain modular forms. Univ. Bergen Arb. naturv. r. Nr. 19. (1959)
17. Morris, A., Yaseen, K.: Some combinatorial results for shifted Young diagrams. Math. Proc. Camb. Philos. Soc., **99**, 23–31 (1986)
18. Olsson, J.: Frobenius symbols for partitions and degrees of spin characters. Math. Scand. **61**, 223–247 (1987)
19. Ramanujan, S.: Collected Papers of S. Ramanujan. London, New York: Cambridge Univ. Press 1927 (reprinted by Chelsea, New York, 1962)
20. Watson, G.N.: Ramanujans Vermutung über Zerfällungsanzahlen. J. Reine Angew. Math. **179**, pp. 97–128 (1938)
21. Watson, G.N.: Proof of certain identities in combinatory analysis. J. Ind. Math. Soc. **20**, 57–69 (1933)

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Classification of subfactors: the reduction to commuting squares

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Introduction

In [J1] Jones introduced the notion of index of a subfactor of a type II_1 factor. This concept, as well as the results in [J1], proved to be very deep and enlightening for numerous aspects of the theory of type II_1 factors. One of the surprising results in [J1] is the characterization of the possible values of indices of subfactors (less than 4) as being the same as the set of square norms of matrices with nonnegative integer entries (up to 4), that is $4\cos^2\frac{\pi}{n}$, $n \geqq 3$.

Among the operator algebraic problems arising from Jones' work the most important are the characterization of all real values > 4 that may occur as indices of subfactors $N \subset M$ with trivial relative commutant $N' \cap M = \mathbf{C}$ and the classification of subfactors having the same index when the ambient factor M is the hyperfinite II_1 factor. One way to approach these problems is by finite dimensional approximation. This means to construct finite dimensional subalgebras $B_n \subset M$ so that $B_n \uparrow M$, $A_n = B_n \cap N \uparrow N$. In addition the algebras $A_n \subset B_n$ should satisfy the so-called commuting square condition $(*)$ $E_{B_n} E_{A_{n+1}} = E_{A_n}$, E_B denoting the trace preserving conditional expectation onto B. The subfactors in [J1] do have this approximation property and in fact the corresponding sequences $A_n \subset B_n$ are uniquely determined by an initial commuting square

$$\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & A_1 \end{array},$$

the rest of the sequence being obtained by an iterative construction.

We prove in this paper a generating result for pairs of hyperfinite factors $N \subset M$ which will show in particular that all subfactors of index $[M:N]$ less than 4 arise in this way, from an initial commuting square of finite dimensional algebras, with inclusion matrices of norm $[M:N]^{1/2}$ and having dimensions and traces

determined by these matrices. This result reduces the classification of such subfactors to the classification of some finite dimensional commuting squares and this in turn is a purely combinatorial problem. To describe in greater detail our results we'll recall some of the necessary background.

It has been noted by various people (Pimsner-Popa, Jones in 1983, Ocneanu 1984) that there are some (almost) canonical finite dimensional subalgebras $B_n \subset M$, associated to $N \subset M$, that satisfy $(*)$: these are the relative commutants in M of the algebras in the Jones' tunnel of subfactors $M \supset N \supset N_1 \supset N_2 \supset \ldots$, i.e. $B_i = N_i' \cap M$. At each step i, N_i (and thus B_i) are unique only up to unitary conjugacy by elements from the previous step N_{i-1} (cf. [PiPo1]). And the problem was posed on whether one can choose the tunnel N_i such that $N_i' \cap M$ generate M and $N_i' \cap N$ generate N, a result that would reduce the study of $N \subset M$ to a study of finite dimensional commuting squares.

Ocneanu was the first to realize that this generating problem may have a positive solution if $N' \cap M = \mathbf{C}$ and $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$, a condition he called finite depth (in fact an equivalent form of it). Since if $N \subset M \subset M_1 \subset M_2 \subset \ldots$ is the Jones' canonical tower of factors associated to $N \subset M$ then the finite dimensional algebra $N_i' \cap M$ is antiisomorphic (and thus isomorphic) to $M' \cap M_{i+1}$, the condition $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$ is the same as $\sup \dim \mathscr{Z}(M' \cap M_i) < \infty$.

We prove in this paper the following generating result, without $N' \cap M = \mathbf{C}$ being assumed.

Theorem. *Let $N \subset M$ be a pair of hyperfinite II_1 factors with finite index. Assume the Jones' tower of factors $N \subset M \subset M_1 \subset M_2 \subset \ldots$ satisfies* $\sup \dim Z(M' \cap M_i) < \infty$. *Then there is a choice of the tunnel of subfactors $M \supset N \supset N_1 \supset N_2 \ldots$ such that $N_i' \cap M \uparrow M$, $N_i' \cap N \uparrow N$. Moreover if $\tilde{M} = \overline{\bigcup_i M_i}$ then the pairs $N \subset M$ and $M'_1 \cap \tilde{M} \subset M' \cap \tilde{M}$ are antiisomorphic.*

The case $N' \cap M = \mathbf{C}$ of this result has been announced at various conferences during 1987–1988 by Ocneanu ([Oc]), however without presenting a proof since then.

The above theorem reduces the classification of subfactors with the property $\sup \dim \mathscr{Z}(M' \cap M_i) < \infty$ to the classification of sequences $\{N_i' \cap N \subset N_i' \cap M\}_i$. But these sequences are uniquely determined by an initial commuting square

$$\begin{array}{ccccc} N_{i_0}' & \cap & M & \subset & N_{i_0+1}' \cap M, \\ & \cup & & & \cup \\ N_{i_0}' & \cap & N & \subset & N_{i_0+1}' \cap N, \end{array}$$

for some i_0 large enough, with i_0, the inclusions, the dimensions and the traces determined by some integral matrices of norm $[M:N]^{1/2}$. Moreover, since $[M:N] < 4$ implies the finite depth condition, we get a complete classification of subfactors of small index in terms of commuting squares:

Theorem. *The conjugacy class of a subfactor $N \subset M$ with finite depth of the hyperfinite II_1 factor M is uniquely determined by the isomorphism class of a*

commuting square of finite dimensional algebras canonically associated to the inclusion $N \subset M$. Moreover, the subfactors of index < 4 automatically satisfy the finite depth condition, so they are all classified by their corresponding commuting squares.

So the problem remains to classify the corresponding commuting squares. In case the number is small (for instance $< 2 + \sqrt{5}$) their number can actually be estimated. Moreover, in the case $[M:N] < 4$ the analysis of the commuting squares has to do only with the simple matrices corresponding to the Dynkin diagrams A_n, D_n, E_6, E_7, E_8, (these are all possible matrices of norm < 2). One can actually obtain:

Corollary. *For each $n \geqq 3$, there are only finitely many subfactors (up to conjugacy) of the hyperfinite II_1 factor, with index $4\cos^2\frac{\pi}{n}$.*

In [Oc], Ocneanu gave an intrinsic characterization of the sequences $\{N_i' \cap N \subset N_i' \cap M\}_i$ in case $N' \cap M = \mathbf{C}$, describing them as group-like objects and announced a full list of such objects for $[M:N] < 4$. Thus a complete list of the subfactors of index < 4 of the hyperfinite II_1 factor can actually be given.

An important application of the case $N' \cap M \neq \mathbf{C}$ of the theorem is the classification up to outer conjugacy of (not necessarily outer) actions of finite groups G on the hyperfinite II_1 factor R and more generally of the classification up to conjugacy of the subgroups isomorphic to G in Aut R/int R (i.e. Jones' theorem in [J3]): this follows by taking a partition of the unity with $|G|$ projections p_g of equal trace in M, $R = p_e M p_e$ and by taking $N = \{\oplus_{g \in G} g(x) | x \in R = p_e M p_e\}$, where $g \in$ Aut R is a lifting of $G \subset$ Aut R/Int R. This application will be discussed in a more general setting in a sequel to this paper.

The proof of the theorem is based on the key observation that if $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$ then the algebras $B = N_i \vee (N_i' \cap M)$ satisfy $E_B(x) \geqq cx$, $x \in M_+$, with a constant $c > 0$ independent on i (see 3.8, 4.3). Since N_i are all hyperfinite (cf. [C]), $N_i \vee (N_i' \cap M)$ can be approximated by relative commutants $N_k' \cap M$, $k > i$ (see 4.4). Thus, by using the inequality one can choose recursively the tunnel N_i such that $R = \overline{\bigcup_i (N_i' \cap M)}$ satisfies $E_R(x) \geqq cx$, $x \in M_+$. By [PiPo1] this means R has finite index in M, $[M:R] \leqq c^{-1}$. The inequality also shows that given $x \in M_+$ and a choice of N_i up to j, there is a choice of N_i, $j < i \leqq k$, so that $N_k' \cap M$ contains a certain "percentage" of x. If this would be true for arbitrary x then the resulting R would equal M. If not, there would be $x \in M$ and j so that $x \perp uRu^*$ for all unitary elements u in N_j (4.6). But by the Rohlin type lemma in [Po4], used here in a form we call the linearization principle (5.1), one has span $uRu^* =$ span $N_j R N_j = M$. Thus $x = 0$ and the theorem is proved (4.9). Then Section 6 contains applications of the main theorem to the classification of subfactors.

The uniform bound from below $E_{N_i \vee (M_i' \cap M)}(x) \geqq cx$, $x \in M_+$, $i \geqq 0$, can be deduced from [Po3]. We included a complete and different proof here for the reader's convenience. The proof is a consequence of the uniform bound for local indices obtained in Section 3, which in turn follows by the general results on sequences of commuting squares in Section 2. Part of the algebraic considerations in Section 3 are of a similar type to some work in [W], [GHJ].

The proof of the main theorem in this paper was presented at the Annapolis Conference on "Knots and Operator Algebras" May 1988 and at the A.M.S. Summer Institute on "Operator Theory and Operator Algebras" August 1988.

1. Preliminaries

1.1. Jones' basic construction and the tower of factors. Let M be a finite von Neumann algebra with a normal finite faithful trace τ, $\tau(1) = 1$. Denote $\|x\|_2 = \tau(x^*x)^{1/2}$, for $x \in M$, and $L^2(M, \tau)$ the completion of M in the norm $\| \ \|_2$.

If $B \subset M$ is a von Neumann subalgebra of M then we denote by E_B the unique τ-preserving conditional expectation of M onto B. Also, we denote by e_B the orthogonal projection of $L^2(M, \tau)$ onto $L^2(B, \tau) \subset L^2(M, \tau)$. E_B, the projection e_B and the canonical conjugation J on $L^2(M, \tau)$, defined by $Jx = x^*$, are related by the properties:

1.1.1 (i) If $x \in M$ then $x \in B$ iff $[x, e_B] = 0$.
(ii) $e_B x e_B = E_B(x)e_B$, $x \in M$.
(iii) $[J, e_B] = 0$.

If M is a type II_1 factor and $N = B \subset M$ is a subfactor then the Jones' index of N in M, $[M:N]$, is the Murray and von Neumann coupling constant of N in its representation by left multiplication on $L^2(M, \tau)$. Thus, the finiteness of $[M:N]$ means that N' (or equivalently $JN'J$) is finite, and its value can then be recuperated as $\tau(e_N)^{-1}$, τ being the (unique) normalized trace on $JN'J$. The construction of the finite factor $JN'J$ is in fact a major tool in studying subfactors: it is called the Jones' *basic construction.* We list now some of its properties:

1.1.2. Assume $[M:N] < \infty$. Define $M_1 = JN'J$. Then M_1 is a type II_1 factor and we have:

(i) $M_1 = (M \cup \{e_N\})'' = \mathrm{span}\{xe_N y, x, y \in M\}$.
(ii) $[M_1:M] = [M:N]$.
(iii) $\tau(e_N x) = [M:N]^{-1}\tau(x)$, $x \in M$, τ being the unique trace on M_1 with $\tau(1) = 1$.

Thus, starting from the initial inclusion $N \subset M$, the basic construction builds a new inclusion $M \subset M_1$, with the same index, with a projection $e_1 = e_N$ of trace $[\mathrm{M:N}]^{-1}$ implementing the conditional expectation (like in 1.1.1). We call e_1 the Jones' projection corresponding to $N \subset M$ and write the basic construction $M \subset^{e_1} M_1$.

We mention that in fact M_1 can be described abstractly: it is the unique II_1 factor that contains M and a projection e_1 so that M and e_1 generate M_1 and so that $e_1 x e_1 = E_N(x)e_1$, $x \in M$ (cf. [PiPo2]).

By iterating this construction we get a whole *tower of factors* $N \subset M \subset M_1 \subset M_2 \subset, \ldots,$ with projections $e_1, e_2, \ldots,$ so that $M_i \subset^{e_{i+1}} M_{i+1}$ is the basic construction for $M_{i-1} \subset M_i$. They satisfy:

1.1.3. (i) $e_{i+1}xe_{i+1} = E_{M_{i-1}}(x)e_{i+1}$, in particular $e_{i+1}e_ie_{i+1} = [M:N]^{-1}e_{i+1}$. Also, $e_ie_{i+1}e_i = [M:N]^{-1}e_i$.

(ii) $e_{i+1} \in M_{i-1}' \cap M_{i+1}$. In particular $e_ie_j = e_je_i$, for $|i-j| \geq 2$.

(iii) $\tau(e_{i+1}x) = [M:N]^{-1}\tau(x)$, $x \in M_i$. In particular $\tau(e_{i+1}w) = [M:N]^{-1}\tau(w)$, for $w \in \mathrm{Alg}\{1, e_1, e_2, \ldots, e_i\}$.

1.2. The finite dimensional case. If $N \subset M$ are finite dimensional then we can still define $M_1 = JN'J$, which will be finite dimensional as well and will satisfy properties similar to 1.1:

1.2.1. $M_1 = (M \cup \{e_i\})'' = \mathrm{span}\{xe_Ny | x, y \in M\}$.

Moreover if $\tilde{M}$ is any algebra containing a copy of M and a nonzero projection e satisfying:

(a) $[N, e] = 0$;
(b) $x \in N$, $xe = 0$ implies $x = 0$;
(c) $exe = E_N(x)e$

then $\Sigma xe_Ny \mapsto \Sigma xey$ defines an isomorphism of M_1 onto the algebra $\mathrm{span}\{xey | x, y \in M \subset \tilde{M}\}$. This observation gives an abstract characterization of M_1 (see. e.g. [W]). We call $M \subset M_1$ with the projection e_1 the *algebraic basic construction* for $N \subset M$ and E_N.

By the definition of M_1 there is a natural isomorphism $z \mapsto JzJ$ between the centers of N and M_1. It can be alternatively described by: "JzJ is the unique element $z_1 \in \mathscr{Z}(M_1)$ satisfying $z_1e_N = ze_N$". This gives a natural identification between the sets of simple summands of N and M_1. If we denote by K this common set and by $A = (a_{kl})_{\substack{k \in K \\ l \in L}}$ the inclusion matrix of $N \subset M$ then the inclusion matrix of $M \subset M_1$ is A^t.

The case when M_1 has a trace which extends the trace of M and satisfies a property similar to 1.1. was clarified in [J1]:

1.2.2. Let $(s_k)_{k \in K}$ be the traces of the minimal projections of N and $(t_l)_{l \in L}$ those of M. Then the following conditions are equivalent.

1°. There is a trace τ_1 on M_1 such that $\tau_1|_M = \tau$ and $\tau_1(xe_N) = \tau_1(x)\tau_1(e_N)$, $x \in M$.

2°. $AA^t(s_k) = \lambda^{-1}(s_k)$ for some $\lambda > 0$.

3°. $A^tA(t_l) = \lambda^{-1}(t_l)$ for some $\lambda > 0$.

Moreover if the conditions hold true then the scalars λ of 2° and 3° satisfy $\lambda = \tau_1(e_N) = \|A\|^{-2}$, all irreducible components A_0 of A satisfy $\|A_0\| = \|A\|$ and (s_k) (resp. (t_l)) is a Perron-Frobenius eigenvector for AA^t (resp. A^tA).

Property 1° for the trace on M_1 is called the $\lambda(= \tau_1(e_N))$ Markov property. If the above conditions are satisfied then one can iterate the basic construction getting at each step the appropriate extension of traces satisfying the Markov property.

If $M \subset M_1$ satisfies 1.2.2 we simply call it the *basic construction* and write $M \subset^{e_N} M_1$.

1.3. The tunnel of subfactors. Jones also showed that given an inclusion of factors with finite index $N \subset M$, there is a subfactor $N_1 \subset N$ and a projection $e_0 \in M$ so that $N \subset {}^{e_0}M$ is the basic construction for $N_1 \subset N$. This fact is made more precise by the results in [PiPo1]. There is a unique (up to conjugacy by unitary elements $u \in N$) projection $e_0 \in M$ such that $E_N(e_0) = [M:N]^{-1}1$. And if $N_1 = \{e_0\}' \cap N$ then N_1 is a subfactor in N with the property that $N \subset {}^{e_0}M$ is the basic construction for $N_1 \subset N$ (i.e. with e_0 satisfying $e_0 x e_0 = E_{N_1}(x)e_0$, $x \in N$). In particular N_1 is unique up to conjugacy by unitary elements in N.

As before, we may iterate this construction k times and get a *tunnel of subfactors* up to k, $M \supset {}^{e_0}N \supset {}^{e_{-1}}N_1 \supset {}^{e_{-2}}N_2 \supset {}^{e_{-3}} \ldots \supset {}^{e_{-k+1}}N_{k-1} \supset N_k$, with $[N_i:N_{i+1}] = [M:N]$, $e_{-i}xe_{-i} = E_{N_{i+1}}(x)e_{-i}$, $x \in N_i$, and $\tau(e_{-i}) = [M:N]^{-1}$, $0 \leqq i \leqq k-1$.

Note however that while the tower $N \subset M \subset {}^{e_1}M_1 \subset {}^{e_2}M_2 \subset$ (with the corresponding projections) is canonical and unique, the tunnel of subfactors

$$M \supset {}^{e_0}N \supset {}^{e_{-1}}N_1 \supset {}^{e_{-2}}N_2 \supset \ldots$$

is unique only up to conjugacy by unitary elements, i.e. if $N_1, N_2, \ldots, N_k$ and $e_0, e_{-1}, \ldots, e_{-k+1}$ are chosen then $e_{-k} \in N_{k-1}$ and $N_{k+1} \subset N_k$ are unique up to conjugacy by a unitary element in N_k.

1.4. Other characterizations of $[M:N]$. There are two alternative descriptions of $[M:N]$ that have been given in [PiPo1]: $[M:N]^{-1}$ is the best constant λ for which the *basic inequality*, $E_N(x) \geqq \lambda x$, holds true for all $x \in M_+$. More precisely, if

$$\lambda(M, N) = \max\{\lambda \geqq 0 | E_N(x) \geqq \lambda x, x \in M_+\}$$

then $\lambda(M, N) = [M:N]^{-1}$. In addition, we saw in 1.3 that there is a projection e (unique up to unitary conjugacy by elements in N) such that $E_N(e) = \lambda(M, N)$. The basic inequality then shows that $\lambda(M, N)$ is the minimal scalar that can be the expected value on N of a nonzero projection in M (see more on this in [Po3]).

Moreover, the module dimension definition of $[M:N]$ can be made more precise: it was shown in [PiPo1] that there exists $\{m_i\}_i \subset M$, a finite set, such that $E_N(m_i^* m_i) = \delta_{ij} f_j$, with $f_j \in \mathcal{P}(N)$, $M = \sum_i m_i N$. A set $\{m_i\}_i$ like this is called an *orthonormal basis* of M over N. Any such set satisfies $\sum_i m_i m_i^* = [M:N] \cdot 1$. The projections f_j can be chosen to be all but possibly one equal to the identity.

Such an orthonormal basis obviously exists for finite dimensional pairs $N \subset M$ (and E_N) as well. It can be used to prove the abstract characterization of the basic construction in 1.2. Moreover if the conditions in 1.2.2 are satisfied then $\Sigma m_i m_i^* = \lambda^{-1}$ (λ as in 1.2.2).

2. Sequences of commuting squares

If M is isomorphic to the hyperfinite II_1 factor then a natural way to approach index problems for subfactors $N \subset M$ is by approximating the pair $N \subset M$ with pairs of finite dimensional algebras $B_n \subset M$ such that $B_n \uparrow M$, $B_n \cap N \uparrow N$ (see

[PiPo1, 3]). The natural condition to be imposed on the algebras B_n is then:

$$E_{B_n}E_N = E_{B_n \cap N} \qquad (*)$$

This is called the *commuting square condition* ([Po1], [PiPo1], [Po2]). Note that if $B \subset M$ satisfies (*) then $\max\{\lambda \geqq 0 | E_{B\cap N}(x) \geqq \lambda x, x \in B_+\} \geqq [M:N]^{-1}$ (cf. [Po2]).

We will prove in this section some general properties for sequences of finite dimensional subalgebras satisfying the commuting square conditions with the Jones tower (or tunnel) of factors. We first consider two consecutive commuting squares.

2.1 Proposition. *Let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2$ be two steps of the tower. Let $B_0 \subset M$, $B_1 \subset M_1$, $B_2 \subset M_2$ be finite dimensional subalgebras satisfying $E_{B_i}E_{M_{i-1}} = E_{B_{i-1}}$, $i = 1, 2$. Suppose $e_2 \in B_2$ and denote by p_2 the central support of e_2 in B_2. Then we have:*

1°. *$B_1 \ni x \mapsto xp_2 \in B_1 p_2$ is an isomorphism;*

2°. *$e_2B_2e_2 = B_0e_2$ and $B_2p_2 = \mathrm{Alg}(B_1p_2 \cup \{e_2\}) = \mathrm{span}\, B_1e_2B_1$;*

3°. *$B_1 \simeq B_1p_2 \subset^{e_2} B_2p_2$ is the algebraic basic construction corresponding to $B_0 \subset B_1$;*

4°. *If we identify the sets of simple summands of B_0 and B_2p_2 via the equality $B_0e_2 = e_2B_2e_2$ (see 1.2.1) then the inclusion matrix for $B_1 \subsetneq B_2p_2$ is the transpose of the inclusion matrix for $B_0 \subset B_1$;*

5°. *If $(s_k)_k$ is the vector giving the traces of the minimal projections of B_0 then the traces of the minimal projections of B_2p_2 are given by $(\lambda s_k)_k$.*

Proof. Let $q_0 = \vee\{ue_2u^* | u \in \mathcal{U}(B_1)\}$. Note that $e_2 \leqq q_0 \leqq p_2$, $q_0 \in B_1' \cap B_2$ and that $B_1 \ni x \mapsto xq_0 \in B_1q_0$ in an isomorphism. Thus, since $e_2xe_2 = E_{B_0}(x)e_2$, $x \in B_1$, it follows that if we denote $B_2^- = \mathrm{span}\, B_1e_2B_1$ then the inclusion $B_1q_0 \subset^{e_2} B_2^-(= B_2^-q_0)$ is the algebraic basic construction for $B_0 \subset B_1$ (cf. 1.2). In particular the inclusion matrix for $B_1q_0 \subset B_2^-$ is the transpose of the inclusion matrix for $B_0 \subset B_1$.

We now show that $q_0 = p_2$ and that $B_2^- = B_2p_2$ and this will end the proof.

To do this we first show that $p_2 = q_0$. We have $q_0 = p \vee \{l(xe_2) | x \in B_1\}$ and $p_2 = \vee\{l(ye_2) | y \in B_2\}$, where $l(z)$ denotes the left support of an element z. But by [PiPo1] and by the commuting square condition, if $x = \lambda^{-1}E_{B_1}(ye_2)$ for some $y \in B_2$ then $x = \lambda^{-1}E_{M_1}(ye_2)$ and $xe_2 = ye_2$. Thus $l(xe_2) = l(ye_2)$ and $q_0 = p_2$. To prove that B_2^- (= span $B_1e_2B_1$) equals B_2p_2 it suffices to show that $e_2B_2e_2 = B_0e_2$. Indeed, since $p_2 = q_0 = \sum_i m_ie_2m_i^*$ for some $m_i \in B_1$, with $E_{B_0}(m_i^*m_j) = \delta_{ij}f_j, f_j \in \mathcal{P}(B_0)$ it would then follow that

$$B_2p_2 = p_2B_2p_2 = q_0B_2q_0 \subset \sum_{i,j} m_ie_2B_2e_2m_j^*$$

$$= \sum_{i,j} m_iB_0e_2m_j^* \subset \mathrm{span}\, B_1e_2B_1 \subset B_2p_2\,.$$

But $e_2B_2e_2 \subset e_2M_2e_2 = Me_2$, so that $e_2B_2e_2 = Xe_2$ for some $X \subset M$. Thus $X = \lambda^{-1}E_{M_1}(Xe_2) = \lambda^{-1}E_{M_1}(e_2B_2e_2) \subset B_1$, so that $X \subset B_1 \cap M = B_0$. This shows that $e_2B_2e_2 \subset B_0e_2 \subset e_2B_2e_2$, ending the proof of the proposition. Q.E.D.

From 2.1 we can now deduce the following property for a sequence of commuting squares:

2.2 Corollary. *Let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ be the Jones' tower of factors and $B_i \subset M_i$, $i \geqq 0$, be finite dimensional subalgebras satisfying $E_{B_i}E_{M_{i-1}} = E_{B_{i-1}}$, $i \geqq 1$ and $e_i \in B_i$, for all $2 \leqq i < n$ (where $2 < n \leqq \infty$). Let K_i (respectively L_i) denote the sets of simple summands of B_{2i} (respectively B_{2i+1}), $n > i \geqq 0$, and identify K_i (resp. L_i) with a subset of K_{i+1} (resp. L_{i+1}) by using 2.1. Denote $K = \bigcup_i K_i$, $L = \bigcup_i L_i$. There exists a unique matrix of nonnegative integers $A = (a_{kl})_{\substack{k\in K\\ l\in L}}$ with the properties.*

1°. $(a_{kl})_{\substack{k\in K_i\\ l\in L_i}}$ describes the inclusion $B_{2i} \subset B_{2i+1}$, for $i \geqq 0$, $2i + 1 < n$;

2°. $(a_{kl})_{\substack{k\in K_{i+1}\\ l\in L_i}}^{t}$ describes $B_{2i+1} \subset B_{2i+2}$, $i \geqq 0$, $2i + 2 < n$;

3°. K_i, L_i satisfy $L_i = \{l \in L \mid \exists k \in K_i \text{ with } a_{kl} \neq 0\}$ and $K_{i+1} = \{k \in K \mid \exists l \in L_i \text{ with } a_{kl} \neq 0\}$.

4°. $\|A\|^2 \leqq [M:N]$ and A is irreducible iff $(a_{kl})_{k\in K_0,\, l\in L_0}$ is.

Proof. The construction of K_i, L_i, K, L and $(a_{kl})_{kl}$, with 1°, 2°, follows by induction from 2.1. To prove 3°, let p', p'', q' be minimal central projections of B_{i+1}, B_{i+2}, B_{i-1} respectively, with $e_{i+1}p' = e_{i+1}q'$ and $p'p'' \neq 0$. We must show there exists some minimal central projection $q'' \in B_i$ such that $e_{i+2}p'' = e_{i+2}q''$ and $q''q' \neq 0$. Define $q'' = \lambda^{-1}E_{B_{i+1}}(p''e_{i+2})$. Since $e_{i+2}B_{i+2}e_{i+2} = B_ie_{i+2}$, q'' is a minimal central projection in B_i. Also $E_{B_{i+1}}(p'')p' = \alpha p'$ for some $\alpha \neq 0$. Thus $q'q'' = \lambda^{-1}q'E_{B_{i+1}}(p''e_{i+2})$ so that

$$p'e_{i+1}q'q''e_{i+1}p' = \lambda^{-1}p'q'E_{B_{i+1}}(e_{i+1}p''e_{i+2}e_{i+1}) = p'q'E_{B_{i+1}}(e_{i+1}p'')$$
$$= p'(q'e_{i+1})E_{B_{i+1}}(p'') = \alpha p'e_{i+1} \neq 0\,.$$

Thus $q'q'' \neq 0$.

The irreducibility condition for A follows by 3°. Moreover by 2.9 in [PiPo3] we have $\|(a_{kl})_{\substack{k\in K_n\\ l\in L_n}}\|^2 \leqq [M:N]$ for all n, so that $\|A\|^2 = \lim_n \|(a_{kl})_{\substack{k\in K_n\\ l\in L_n}}\|^2 \leqq [M:N]$. Q.E.D.

2.3 Corollary. *If $\{B_i\}_{n>i\geqq 0}$ are finite dimensional subalgebras ($n \leqq \infty$) with $B_i \subset M_i$, for $i \geqq 0$, $E_{B_i}E_{M_{i-1}} = E_{B_{i-1}}$, for $i \geqq 1$, $e_i \in B_i$, for $i \geqq 2$, and if there exists $i_0 \geqq 0$ such that $\dim \mathscr{Z}(B_{i_0+2}) \leqq \dim \mathscr{Z}(B_{i_0})$, then $\{B_i\}_{n>i\geqq i_0}$ with the projections $e_i \in B_i$, $i \geqq i_0 + 2$, are obtained by iterating the basic construction starting from the inclusion $B_{i_0} \subset B_{i_0+1}$ and at each step the unique trace τ of $\cup M_i$ is a $\lambda = [M:N]^{-1}$-Markov trace. Thus, if $A = (a_{kl})_{kl}$ is the inclusion matrix of $B_{i_0} \subset B_{i_0+1}$ and $(s_k)_k$ are the traces of the minimal projections of B_{i_0} and $(t_l)_l$ those of B_{i_0+1} then*

1°. The inclusions $B_{i_0} \subset B_{i_0+1} \subset B_{i_0+2} \ldots$ are given by A, A^t, A, ...

2°. The traces of minimal projections of B_{i_0+2j} are given by $(\lambda^j s_k)_k$ and those of B_{i_0+2j+1} by $(\lambda^j t_l)_l$.

3°. $\|A\|^2 = [M:N] = \lambda^{-1}$ and $\|A_0\|^2 = [M:N]$ for any irreducible component A_0 of A.

4°. $(s_k)_k$ is a Perron-Frobenius eigenvector for AA^t and $(t_l)_l$ a Perron-Frobenius eigenvector for A^tA.

Proof. Trivial by 2.1, 2.2, 1.2. Q.E.D.

As 2.3 shows, the existence of a sequence of commuting squares satisfying the hypothesis of 2.3 imposes restrictions on the index of N in M: we must have $[M:N] = \|A\|^2$. We now deduce that we "almost must have" $N' \cap M = C$.

2.4 Corollary. *Assume there exists a sequence of finite dimensional subalgebras $B_i \subset M_i$, $i \geq 0$, satisfying the hypothesis of* 2.3. *Then $E_{N' \cap M}(e_0) = [M:N]^{-1} 1$ for all Jones projections $e_0 \in M$ (i.e., projections e_0 for which $E_N(e_0) = [M:N]^{-1} 1$).*

Proof. Let $B_{i_0} = \oplus_k M_{n_k \times n_k}(\mathbf{C})$. Then by [PiPo1] and 2.3 (see also [PiPo3]) we have

$$H(B_{i_0+2j+1} | B_{i_0+2j}) = \sum_{k,l} (a_{kl}(AA^t)^j \vec{n})_k t_l \lambda^j \ln \frac{(A^t(AA^t)^j \vec{n})_l (A\lambda^j \vec{t})_k}{((AA^t)^j \vec{n})_k (\lambda^j t_l)} .$$

Since $(AA^t)^j \vec{n}$ tends (up to renormalization) to the Perron-Frobenius eigenvector of AA^t, i.e. to a scalar multiple of $A\vec{t}$, it follows that

$$\lim_{j \to \infty} \frac{(A^t(AA^t)^j \vec{n})_l (A\vec{t})_k}{((AA^t)^j \vec{n})_k t_l} = \lambda^{-1} .$$

Thus, since $\sum_{k,l} (a_{kl}(AA^t)^j \vec{n})_k t_l \lambda^j = 1$, it follows that $\lim_{j \to \infty} H(B_{i_0+2j+1} | B_{i_0+2j}) = \ln \lambda^{-1}$. Since $M_{i_0} \subset M_{i_0+1}$ is isomorphic to a reduction of the pair $M_{i_0+2j} \subset M_{i_0+2j+1}$ by a projection in M_{i_0+2j} (cf. [PiPo1]) it follows that $H(M_{i_0+2j+1} | M_{i_0+2j}) = H(M_{i_0+1} | M_{i_0})$ for all $j \geq 0$ (cf. e.g. 4.4 in [PiPo1]). Thus from the commuting square condition we get

$$H(B_{i_0+2j+1} | B_{i_0+2j}) \leq H(M_{i_0+2j+1} | M_{i_0+2j}) = H(M_{i_0+1} | M_{i_0}) \leq \ln \lambda^{-1}$$

which shows that $H(M_{i_0+1} | M_{i_0}) = \ln \lambda^{-1}$. By [PiPo2] this implies $H(M|N) = \ln \lambda^{-1}$ and $E_{N' \cap M}(e_0) = [M:N]^{-1} 1$. Q.E.D.

If B_i, $i \geq 0$, satisfy the conditions of 2.2, then $e_i \in B_i$ for all $i \geq 2$ implies that $\overline{\bigcup_i B_i^w} \subset \overline{\bigcup_i M_i^w} = \tilde{M}$ is a type II_1 von Neumann algebra (since $\{e_i\}_i''$ is a type II_1 factor). In fact $\overline{\bigcup_i B_i^w}$ follows to be a factor under very general assumptions. We will prove this type of results elsewhere. For what we need the next easy consequence of 2.3 will do.

2.5 Corollary. *Let $\{B_i\}_{i \geq 0}$ be like in* 2.3 *(with $n = \infty$), and assume the inclusion matrix of $B_0 \subset B_1$ is irreducible. Then $R = \overline{\bigcup_i B_i^w}$ is a type II_1 subfactor in $\overline{\bigcup_i M_i^w} = \tilde{M}$.*

Proof. Let $e^{k,i}$, $f^{l,i}$ be the minimal central projections of B_{2i} respectively B_{2i+1}. Arguing like in 2.4 (see also [PiPo3]) it follows that $1°$. $\lim_{i \to \infty} \tau(e^{k,i}) = v_k^2$; $2°$. $\lim_{i \to \infty} \tau(f^{l,i}) = w_l^2$; and $3°$. $\lim_{i \to \infty} \tau(e^{k,i} f^{l,i}) = \lambda^{1/2} a_{kl} v_k w_l$, where $(v_k)_k$, $(w_l)_l$ are the Perron-Frobenius eigenvectors of AA^t, respectively A^tA normalized so that $\sum_k v_k^2 = 1$, $\sum_l w_l^2 = 1$, and $\lambda = \|A\|^{-2}$. Let $p \in \mathscr{P}(\mathscr{Z}(R))$ and $p_n \in \mathscr{P}(\mathscr{Z}(B_n))$ such that $\|p_n - p\|_2 \to 0$. In particular we have: $4°$. $\|p_{2n} p_{2n+1} - p_{2n}\|_2 \to 0$. Let $E_n \subset K$, $F_n \subset L$ with $p_{2n} = \sum_{k \in E_n} e^{k,n}$, $p_{2n+1} = \sum_{l \in F_n} f^{l,n}$. Then $3°$, $4°$ show that, for

n large enough, if $e^{k,n}f^{l,n} \neq 0$ (or equivalently $a_{kl} \neq 0$) for some $k \in E_n$, $l \in L$, then $l \in F_n$. Thus

$$F_n = \{l \in L \mid a_{kl} \neq 0 \text{ for some } k \in E_n\}$$

and $p_{2n+1} \geqq p_{2n}$, so in fact $p_{2n+1} = p_{2n}$ again by 3°, 4°. But this contradicts the irreducibility of A, unless $E_n = K$, $F_n = L$, $p_{2n} = 1$. Q.E.D.

We will refer to the property $\sup \dim \mathscr{Z}(B_n) < \infty$ for an increasing sequence of finite dimensional algebras $\{B_n\}_n$ by saying that it has *bounded* (or *finite*) *growth*.

3. Higher relative commutants and finite depth

Approximating a pair of hyperfinite factors $N \subset M$ with finite dimensional subalgebras satisfying the commuting square condition (*) is in fact quite difficult. However, there is a simple way of producing finite dimensional subalgebras of M satisfying (*) by taking the relative commutants in M of some subfactors $\tilde{N}$ of N. Indeed by [Po1] we then have $E_{\tilde{N}' \cap M} E_N = E_{\tilde{N} \cap N}$. Also, if $[N:\tilde{N}] < \infty$ then $\tilde{N}' \cap M$ is finite dimensional. A possible choice for $\tilde{N}$ are the factors in the tunnel.

3.1 Proposition. *Let* $M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} N_2 \supset \ldots$ (*respectively* $M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$) *be a choice of the tunnel of factors* (*resp. the Jones tower*). *Then we have*

1°. $N_j' \cap M$, $M' \cap M_j$ *are finite dimensional.*

2°. $E_{N_j' \cap M} E_{N_i} = E_{N_j' \cap N_i}$, $E_{M_k' \cap M_j} E_{M_i} = E_{M_k' \cap M_i}$, *for* $k \leqq i \leqq j$.

3°. $e_0, e_{-1}, \ldots, e_{-i+1} \in N_i' \cap M$, $e_2, e_3, \ldots, e_j \in M' \cap M_j$.

Proof. By [J1] $N_i' \cap M_i$ are finite dimensional and by [Po1] we have the corresponding commuting squares. Q.E.D.

Although the tunnel of factors N_k is not uniquely determined (see 1.3) the resulting pair of algebras $\overline{\bigcup_i (N_i' \cap N)} \subset \overline{\bigcup_i (N_i' \cap M)}$ is actually unique up to isomorphism.

3.2 Proposition. 1°. *Let* $M \supset^{e_0} N \supset^{e_{-1}} N_1 \supset^{e_{-2}} N_2 \supset \ldots$, $M \supset^{e_0^0} N \supset^{e_{-1}^0} N_1^0 \supset^{e_{-2}^0} N_2^0 \supset \ldots$, *be two choices of the tunnel. Denote* $R = \overline{\cup (N_j' \cap M)^w}$, $R_0 = \overline{\bigcup_j (N_j' \cap N)^w} \subset R$, $R^0 = \overline{\bigcup_j (N_j^{0\prime} \cap M)^w}$, $R_0^0 = \overline{\bigcup_j (N_j^{0\prime} \cap N)^w} \subset R^0$. *Then there exists a trace presenting isomorphism of* R *onto* R^0 *carrying* R_0 *onto* R_0^0 *and* $N_j' \cap M$ *onto* $M' \cap M_{j+1, j \geqq 0}$.

2°. *Let* $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ *be the tower of factors and* $\tilde{M} = \overline{\bigcup_n M_n}$, $R^1 = M' \cap \tilde{M}$, $R_0^1 = M_1' \cap \tilde{M}$. *Then there exists an antiisomorphism* $\Phi: \bigcup_j (N_j' \cap M) \to \bigcup_j (M' \cap M_{j+1})$ *with* $\Phi(N_j' \cap M) = M' \cap M_{j+1}$, $\Phi(N_j' \cap N) = M_1' \cap M_{j+1}$, $\Phi(e_{-j}) = e_{j+2}$. *Moreover if* $N \subset M$ *satisfies* $E_{N' \cap M}(e_0) = [M:N]^{-1} 1$ *then* Φ *is trace preserving and can be uniquely extended to a trace preserving antiisomorphism of* R *onto* R^1 *carrying* R_0 *onto* R_0^1.

Proof. 1°. By [PiPo1] (see 1.3) it follows that there exist unitary elements $u_i \in N_i^0$, $i \geqq 0$, such that $u_{i-1} \ldots u_0 N_i u_0^* \ldots u_{i-1}^* = N_i^0$, $i \geqq 1$. Define $\sigma: R \to R^0$ by $\sigma(x) = \lim_{i \to \infty} u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^*$. If $x \in N_i' \cap M$, then

$u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^* \in N_i^{0\prime} \cap M$ so that

$$u_n \ldots u_i (u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^*) u_i^* \ldots u_n^* = u_{i-1} \ldots u_0 x u_0^* \ldots u_{i-1}^* ,$$

for all $n > i-1$. This shows that σ is a well defined trace preserving isomorphism. Obviously $\sigma(R_0) = R_0^0$.

2°. By [PiPo2] one can represent recursively each M_n on $L^2(M,\tau)$. Thus $\cup M_n \subset \mathcal{B}(L^2(M,\tau))$ and $N_i = JM'_{i+1}J$ will form a tunnel of subfactors in M. Since $JM'J = M$ we also have $J(M' \cap (\bigcup_i M_i))J = \bigcup_i (N'_i \cap M)$. Thus the reflection $\Phi(x) = JxJ$ gives the desired antiisomorphism, which, in case $E_{N' \cap M}(e_0) = \lambda$, is trace preserving by [PiPo2] and thus can be extended from $\overline{\bigcup_i (N'_i \cap M)}^w$ onto $M' \cap \tilde{M}$. Q.E.D.

3.3 Definition. The pair of algebras $R_0 = \overline{\bigcup_i (N'_i \cap N)} \subset \overline{\bigcup_i (N'_i \cap M)} = R$ is called the *core* of the inclusion $N \subset M$. Note that the isomorphism class of $R_0 \subset R$ doesn't depend on the choice of the tunnel.

Since the algebras $M' \cap M_i$ satisfy the conditions of 2.2. (cf. 3.1) we can now list the corresponding properties of these algebras:

3.4 Corollary. *Denote $B_i^0 = N'_{i-1} \cap M$ and $B_i = M' \cap M_i$. Then B_i^0 satisfy condition (*) in Section 2 and B_i, $i \geqq 0$, satisfy the hypothesis of 2.2 and they have the properties:*

3.4.1. *If p_{i+2} is the central support of e_{i+2} in B_{i+2} (resp. of e_{-i} in B_{i+2}^0) then $B_{i+2}p_{i+2} = \operatorname{span} B_{i+1}e_{i+2}B_{i+1}$ (resp. $B_{i+2}^0 p_{i+2} = \operatorname{span} B_{i+1}^0 e_{-i} B_{i+1}^0$) and $B_{i+1} \overset{e_{i+2}}{\hookrightarrow} B_{i+2}p_{i+2}$ (resp. $B_{i+1}^0 \simeq B_{i+1}^0 p_{i+2} \subset {}^{e_{-i}}B_{i+2}^0$) is the algebraic basic construction for $B_i \subset B_{i+1}$ (resp. $B_i^0 \subset B_{i+1}^0$).*

3.4.2. *Identify the sets of simple summands of B_{2i} and B_{2i}^0 by 3.2.2 and denote it by K_i, $i \geqq 0$, and those of B_{2i+1}, B_{2i+1}^0 by L_i, $i \geqq 0$. Via these identifications the embeddings $K_i \subset K_{i+1}$ (or $L_i \subset L_{i+1}$) defined in two ways by either the application $\mathcal{Z}(B_i) \ni q' \mapsto q'' \in \mathcal{Z}(B_{i+2}p_{i+2})$, with $q''e_{i+2} = q'e_{i+2}$, or by $\mathcal{Z}(B_i^0) \ni q' \mapsto q'' \in \mathcal{Z}(B_{i+2}^0 p_{i+2})$, with $q''e_{-i} = q'e_{-i}$, are the same.*

3.4.3. *Let $K = \bigcup_i K_i$, $L = \bigcup_i L_i$. Then $K_0 = \{k_0\}$ and there is a unique pointed matrix $A = (a_{kl})_{k \in K, l \in L}$, $k_0 \in K$, such that $(a_{kl})_{\substack{k \in K_i \\ l \in L_i}}$ describes $B_{2i} \subset B_{2i+1}$ and $B_{2i}^0 \subset B_{2i+1}^0$ while $(a_{kl})^t_{\substack{k \in K_{i+1} \\ l \in L_i}}$ describes $B_{2i+1} \subset B_{2i+2}$ and $B_{2i+1}^0 \subset B_{2i+2}^0$. Moreover K_i, L_i satisfy*

$$L_i = \{l \in L \mid \exists k \in K_i \quad \text{with} \quad a_{kl} \neq 0\} ,$$
$$K_{i+1} = \{k \in K \mid \exists l \in L_i \quad \text{with} \quad a_{kl} \neq 0\} .$$

3.4.4. *A is an irreducible matrix and $\|A\|^2 \leqq [M:N]$.*

Proof. cf. 2.2 and 3.1, 3.2.2. Q.E.D.

3.5. Remark. The pointed matrix (A, k_0) associated to the sequences $N'_i \cap M$, $M' \cap M_i$ like in 3.4 coincides with the *standard matrix* defined in [Po3]. It is also equivalent to Ocneanu's principal graph for $N' \cap M = \mathbf{C}$. In fact for what we need in this section the definition and construction of A are not necessary; the result 3.8

which will be used in the proof of the main result in Section 4 can be derived directly from the general result 2.2.

We now mention the particularization to the case $B_i = M' \cap M_i$ of the results on sequences of commuting squares with $\sup \dim \mathscr{Z}(B_i) < \infty$ in Section 2. Following ([Oc]), if the sequence of higher relative commutants of $N \subset M$ has bounded growth, then we say that $N \subset M$ has finite depth. In fact this terminology is used in [Oc] for subfactors with $N' \cap M = \mathbf{C}$. We will use it here for subfactors $N \subset M$ without assuming the trivial relative commutant condition. We also introduce another terminology, which although less established seems better suited.

3.6 Definition. The inclusion $N \subset M$ has *finite depth* (or *finite growth*) if

$$\sup \dim \mathscr{Z}(N_i' \cap M) (= \dim \mathscr{Z}(M' \cap M_{i+1})) < \infty$$

(i.e. if $B_i = M' \cap M_i$ have bounded growth).

3.7 Corollary. *If $N \subset M$ has finite depth then:*

3.7.1. $E_{N' \cap M}(e_0) = [M:N]^{-1}$ 1 *and the antiisomorphism of* 3.2.2 *is trace preserving.*

3.7.2. *There is an i_0 such that for all $i \geqq i_0$, $B_{i+1} \subset^{e_{i+2}} B_{i+2}$, $B_{i+1}^0 \subset^{e_{i+2}} B_{i+2}^0$ are the basic constructions for $B_i \subset B_{i+1}$ respectively $B_i^0 \subset B_{i+1}^0$ and the unique trace τ of $\tilde{M}$ is a $\lambda = [M:N]^{-1}$ Markov trace on each basic construction. Thus the inclusions $B_{2i} \subset B_{2i+1}$, $B_{2i}^0 \subset B_{2i+1}^0$ are described by A and $B_{2i+1} \subset B_{2i+2}$, $B_{2i+1}^0 \subset B_{2i+2}^0$ by A^t, for $2i \geqq i_0$, A being the standard matrix defined in* 3.4.3.

3.7.3. *If p (resp. q) is a minimal projection of B_i^0 or B_i (resp. B_{i+2j}^0 or B_{i+2j}) which belongs to a simple summand labeled by the same k (if i is even) or l (if i is odd) then $\tau(q) = \lambda^j \tau(p)$.*

3.7.4. $\|A\|^2 = [M:N]$ *and for $2i \geqq i_0$ the traces of the minimal projections of B_{2i}, B_{2i}^0 give the Perron-Frobenius eigenvector for AA^t, A being the standard matrix.*

3.7.5. *The algebras $R = \overline{\bigcup_i (N_i' \cap M)}$ and $R \cap N_j = \overline{\bigcup_i (N_i' \cap N_j)}$, $j \geqq 0$, are type II_1 factors.*

Proof. By 2.3, 2.4 and 3.1, 3.2.2 we get 3.7.1–3.7.4. Moreover, by 2.5 and 3.2 it follows that $R = \overline{\bigcup_i (N_i' \cap M)}$ is a factor. Since $[M:N_j] < \infty$ and $E_{N_i' \cap M} E_{N_j} = E_{N_i' \cap N_j}$, by [PiPo3] it follows that the embedding matrix of $N_i' \cap N_j \subset N_i' \cap M$, $i \geqq j \geqq 0$, has square norm majorized by $[M:N]^{j+1}$. Thus, since $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$, $\sup \dim \mathscr{Z}(N_i' \cap N_j) < \infty$ as well. Thus $N_{j+1} \subset N_j$ has finite depth and $R \cap N_j = \overline{\bigcup_i (N_i' \cap N_j)}$ is a factor by the previous part. Q.E.D.

Corollary 3.7 yields an important property for subfactors with finite depth: there is a uniform bound for the local indices $[pMp:N_ip]$ of the inclusions $N_i \subset M$, independent of i and of the minimal projection p of $N_i' \cap M$. More precisely we have:

3.8 Theorem. *Assume $N \subset M$ is a subfactor of finite depth. Let $A = (a_{kl})_{\substack{k \in K \\ l \in L}}$, $k_0 \in K$, be its standard matrix and let $(v_k)_{k \in K}$ be the Perron-Frobenius eigenvector of AA^t*

normalized so that $v_{k_0} = 1$. *Let* $i \geqq 0$ *and* p *be a minimal projection of* $N'_{2i-1} \cap M$ (*or* $M' \cap M_{2i}$) *which belongs to a simple direct summand labeled by* $k \in K$. *Then* $[pMp : N_{2i-1}p] = v_k^2$ (*respectively* $[pM_{2i}p : Mp] = v_k^2$). *Moreover if* q *is a minimal projection in* $N'_{2i} \cap M$ *or* $M' \cap M_{2i+1}$ *in the simple summand labeled by* $l \in L$ *then*

$$[qMq : N_{2i}q] = [M:N]^{-1}\left(\sum_k a_{kl}v_k\right)^2 \Big(\textit{respectively}\, [qM_{2i+1}q : Mq]$$

$$= [M:N]^{-1}\left(\sum_k a_{kl}v_k\right)^2\Big).$$

Proof. Since $E_{N' \cap M}(e_0) = \lambda = [M:N]^{-1}$, by [PiPo2] and 3.1 it is sufficient to prove the statement for the tower. By [PiPo2] we then have $[pM_{2i}p : Mp] = [M_{2i} : M]\tau(p)^2 = \lambda^{-2i}\tau(p)^2$. Let $j_0 \geqq 0$ be so that $M' \cap M_{2j+1} \subset M' \cap M_{2j+2}$ is the basic construction for $M' \cap M_{2j} \subset M' \cap M_{2j+1}$ for $j \geqq j_0$. By 3.7 if $j \geqq 0$ and if p_k^{2j} is a choice of a minimal projection in the k-th summand of $M' \cap M_{2j}$ then $\tau(p_k^{2j}) = \lambda^{j-j_0}\tau(p_k^{2j_0})$. Thus

$$[p_k^{2j}M_{2j}p_k^{2j} : Mp_k^{2j}] = \lambda^{-2j}\tau(p_k^{2j})^2 = \lambda^{-2j_0}\tau(p_k^{2j_0})^2 .$$

But by 3.7 $(\tau(p_k^{2j_0}))_{k \in K}$ is the Perron-Frobenius eigenvector for AA^t normalized by $\tau(p_{k_0}^{2j_0}) = \lambda^{j_0}$. Thus $[p_{k_0}^{2j}M_{2j}p_{k_0}^{2j} : Mp_{k_0}^{2j}] = 1$ and $[p_k^{2j}M_{2j}p_k^{2j} : Mp_k^{2j}]^{1/2}$ coincides with the k-th entry of the Perron-Frobenius eigenvector of AA^t which has the k_0 entry equal to 1.

Similarly if p_l^{2j+1} is a choice of a minimal projection in the l-th summand of $M' \cap M_{2j+1}$, $j \geqq 0$, then $(\tau(p_l^{2j+1}))_l = A^t(\tau(p_k^{2j+2}))_k$ and

$$[p_l^{2j+1}M_{2j+1}p_l^{2j+1} : Mp_l^{2j+1}] = \tau(p_l^{2j+1})^2\lambda^{-2j-1} = \tau(p_l^{2j_0+1})^2\lambda^{-2j_0-1} .$$

Thus we get for $j \geqq j_0$

$$A([p_l^{2j+1}M_{2j+1}p_l^{2j+1} : Mp_l^{2j+1}]^{1/2})_l = \lambda^{-j_0-1/2}(\tau(p_k^{2j_0}))_k$$

$$= \lambda^{-1/2}([p_k^{2j_0}M_{2j_0}p_k^{2j_0} : Mp_k^{2j_0}]^{1/2})_k .$$

This implies that $\lambda^{-1/2}([p_l^{2j+1}M_{2j+1}p_l^{2j+1} : Mp_l^{2j+1}]^{1/2})_l$ is a Perron-Frobenius eigenvector for A^tA so that

$$\lambda^{1/2}A^t([p_k^{2j_0}M_{2j_0}p_k^{2j_0} : Mp_k^{2j_0}]^{1/2})_k = ([p_l^{2j+1}M_{2j+1}p_l^{2j+1} : Mp_l^{2j+1}]^{1/2})_l .$$

This ends the proof of the theorem. Q.E.D.

4. Finite depth and the generating property

As we mentioned before, a general strategy for approaching index problems for subfactors $N \subset M$ of the hyperfinite II_1 factor $M \simeq R$ is to approximate $N \subset M$ by a sequence of finite dimensional subalgebras satisfying the commuting square condition (*). From Section 2 we saw that the finite dimensional subalgebras $B_i^0 = N'_{i-1} \cap M$ are the first to be considered that satisfy $E_{B_i^0}E_N = E_{B_i^0 \cap N}$. Then the approximation with B_i^0 amounts to say that we may choose the tunnel N_i so that the core $R_0 = \overline{\bigcup_i (N'_i \cap N)} \subset \overline{\bigcup_i (N'_i \cap M)} = R$ coincides with $N \subset M$.

Approximating $N \subset M$ with the sequence of higher relative commutants is more rewarding than with other finite dimensional algebras, since the sequence of higher relative commutants is canonical.

4.1 Definition. We say that $N \subset M$ has the *generating property* if there exists a choice of the tunnel of subfactors N_i such that $N_i' \cap M \uparrow M$, $N_i' \cap N \uparrow N$. Note that since $E_{N_i' \cap M} E_N = E_{N_i' \cap N}$, $N_i' \cap M \uparrow M$ actually implies $N_i' \cap N \uparrow N$.

We will prove in this section that if M is the hyperfinite II_1 factor and $N \subset M$ has finite depth (i.e. $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$) then $N \subset M$ has the generating property. The key observation allowing us to prove this generating result is that the subalgebras $B = N_k \vee (N_k' \cap M)$ satisfy the basic inequality $E_B(x) \geqq cx$, $x \in M_+$, with c independent of k. This important device will be a simple consequence of 3.8. But first we need a simple formula which is implicit in [PiPo3].

4.2 Lemma. *Let $n_0, C_0 < \infty$. If $q_1, \ldots, q_n \in \mathscr{P}(M)$ is a partition of the unity in M, with $n \leqq n_0$, and if $P_i \subset q_i M q_i$ are subfactors with $[q_i M q_i : P_i] < C_0$, then $B = \sum_i P_i$ satisfies $E_B(x) \geqq (n_0 C_0)^{-1} x$, $x \in M_+$.*

Proof. Denote $B_1 = \sum_i q_i M q_i$. By [PiPo3] we have $E_{B_1}(x) \geqq n_0^{-1} x$, $x \in M_+$. Moreover if $y \in B_1$, $y = \Sigma y_i$, with $y_i \in q_i M q_i$ then $E_B(y) = \sum_i E_{P_i}(y_i)$. Thus if $y \geqq 0$ then by [PiPo1] (see 1.1) we have $E_B(y) \geqq \sum_i [q_i M q_i : P_i]^{-1} y_i \geqq C_0^{-1} y$. Altogether we get for $x \in M_+$, $E_B(x) = E_B E_{B_1}(x) \geqq C_0^{-1} E_{B_1}(x) \geqq (C_0 n_0)^{-1} x$. Q.E.D.

4.3 Theorem. *Suppose $N \subset M$ has finite depth. There exists $c > 0$ such that given any choice of the tunnel of subfactors $\{N_i\}_{i \geqq 1}$ we have*

$$E_{N_i \vee (N_i' \cap M)}(x) \geqq cx, \; x \in M_+ ,$$

for all $i \geqq 1$.

Proof. Let $\{q_j^i\}_j$ be the minimal central projections of $N_i' \cap M$. Then q_j^i are also the minimal central projections of $N_i \vee (N_i' \cap M)$. Thus $N_i \vee (N_i' \cap M) = \sum_j P_j^i$ with $P_j^i = (N_i \vee (N_i' \cap M)) q_j^i$ subfactors in $q_j^i M q_j^i$. Moreover if p is a minimal projection in $(N_i' \cap M) q_j^i$ then $p \in P_j^i$ and $N_i p = p(N_i \vee (N_i' \cap M)) p = p P_j^i p$ so that

$$[q_j^i M q_j^i : P_j^i] = [pMp : pP_j^i p] = [pMp : N_i p] .$$

By 3.8 there is an upper bound on this indices,

$$\sup \{ [pMp : N_i p] \mid i \geqq 0, \, p \quad \text{minimal projection in } N_i' \cap M \} = C_0 < \infty .$$

Thus $[q_j^i M q_j^i : P_j^i] \leqq C_0$ for all i, j. Moreover, since for each i the number of P_j^i's is bounded by $\sup \dim \mathscr{Z}(N_i' \cap M) = n_0 < \infty$, the statement follows by 4.2. Q.E.D.

If c is the uniform bound (from below) given by 4.3 then the above theorem easily yields the existence of a tunnel N_i so that $R = \overline{\bigcup_i (N_i' \cap M)}^{w}$ has finite index in M. This "almost" proves the generating property. To show all this let us first mention a consequence of the hyperfiniteness of M and of [J1].

4.4 Lemma. *Let $N \subset M$ be hyperfinite II_1 factors with finite index. Let $M \supset^{e_0} N \supset^{e_1} N_1 \supset^{e_2} \ldots \supset^{e_{i-1}} N_{i-1} \supset N_i$ be a choice of the tunnel up to some i. Let $\varepsilon > 0$ and $F \subset N_i \vee (N_i' \cap M)$ a finite set. Then there exists $j \geqq i$ and a*

continuation of the tunnel $\supset^{e-i}N_i \supset^{e-i-1}N_{i+1} \ldots \supset^{e-j+1}N_{j-1} \supset N_j$ *such that* $x \in_\varepsilon N'_j \cap M$, $x \in F$.

Proof. Let $\{f^n_{r,s}\}_{r,s,n}$ be a matrix unit for $N'_i \cap M$, where n labels the summands of $N'_i \cap M$. Then $x = \Sigma x^n_{rs} f^n_{rs}$, with $x^n_{rs} \in N_i$. Since N_i is isomorphic to the hyperfinite II_1 factor (cf. [C]), by [J1] we may regard it as generated by a sequence of projections $\{e'_m\}_{m \geqq 1} \subset N_i$ satisfying Jones' axioms $e'_m e_{m \pm 1} e'_m = \lambda e'_m$, $[e'_m, e'_n] = 0$ for $|m-n| \geqq 2$, $\tau(e'_m) = \lambda$, where $\lambda = [M:N]^{-1}$. Thus for any $\delta > 0$ there is $p \geqq 1$ such that $x^n_{rs} \in_\delta \mathrm{Alg}\{1, e'_1, \ldots, e'_{p-1}\}$. Let $\supset^{e^0_{-i}}N_i \supset^{e^0_{-i-1}} N^0_{i+1} \ldots \supset^{e^0_{-i-p+1}} N^0_{i+p-1} \supset N^0_{i+p}$ be a continuation of the tunnel up to $i+p$. By [J1] there is a τ-preserving isomorphism between $\mathrm{Alg}\{1, e'_1, \ldots, e'_{p-1}\}$ and $\mathrm{Alg}\{1, e^0_{-i-1}, \ldots, e^0_{-i-p+1}\}$ sending e'_k to e^0_{-i-k}. Thus there is a unitary element in the ambient algebra N_i such that $ue^0_{-i-k}u^* = e'_k$, $1 \leqq k \leqq p-1$. Thus, if we put $j = i+p$ and $N_{i+k} = uN^0_{i+k}u^*$, $1 \leqq k \leqq p$, then N_{i+k} is a continuation of the tunnel such that $N'_j \cap M \supset \mathrm{Alg}\{1, e'_1, \ldots, e'_{p-1}\} \ni_\delta x^n_{rs}$ for all n, r, s (and all $x \in F$). Since $f^n_{rs} \in N'_i \cap M \subset N'_j \cap M$ and $x^n_{rs} \in_\delta N'_j \cap M$, if we take δ small enough we get $N'_j \cap M \ni_\varepsilon \Sigma x^n_{rs} f^n_{rs} = x$, $x \in F$. Q.E.D.

4.5 Corollary. *If* $N \subset M$ *has finite depth and* N, M *are hyperfinite* II_1 *factors then there is a choice of the tunnel* $\{N_i\}_{i \geqq 1}$ *such that the factor* $R = \overline{\bigcup_i (N'_i \cap M)}$ *has finite index in* M.

Proof. Let $c > 0$ be the constant in 4.3. Let $\{x_n\}_n \subset M_+$ be a sequence of elements dense in the unit ball of M_+ in the norm $\| \ \|_2$. By 4.3, 4.4 we may choose recursively numbers $k_1 < k_2 < \ldots < k_{n-1} < k_n \ldots$ and continuations of the tunnel $N_{k_{n-1}+1} \supset \ldots \supset N_{k_n}$ such that $\| E_{N'_{k_n} \cap M}(x_i)\|_2^2 \geqq {}_{1/n}c \| x_i\|_2^2$, $1 \leqq i \leqq n$. Indeed, suppose we achieved step $n-1$. By 4.3 we have $y_i \overset{\text{def}}{=} E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)}(x_i) \geqq cx_i$, thus $\| y_i\|_2^2 = \tau(y_i x_i) \geqq c\tau(x_i^2) = c \| x_i\|_2^2$, $1 \leqq i \leqq n$. By 4.4 we may choose $N_{k_{n-1}+1} \supset N_{k_{n-1}+2} \ldots \supset N_{k_n}$, for some $k_n > k_{n-1}$, so that $y_i \in_{1/n} N'_{k_n} \cap M$. Thus, since $N'_{k_n} \cap M$ and $N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)$ form a commuting square (cf. [Po1]), we get $\| E_{N'_{k_n} \cap M}(x_i)\|_2^2 \geqq \| E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)} E_{N'_{k_n} \cap M}(x_i)\|_2^2$
$= \| E_{N'_{k_n} \cap M} E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)}(x_i)\|_2^2 = {}_{1/n}\| E_{N_{k_{n-1}} \vee (N'_{k_{n-1}} \cap M)}(x_i)\|_2^2 \geqq c \| x_i\|_2^2$.
If we take now $R = \overline{\bigcup_{n \geq 1}(N'_{k_n} \cap M)^w} = \overline{\bigcup_{k \geq 1}(N'_k \cap M)}$ then $\| E_R(x_i)\|_2^2 = \lim_n \| E_{N'_{k_n} \cap M}(x_i)\|_2^2 \geqq c \| x_i\|_2^2$ for all i. Thus $\| E_R(x)\|_2^2 \geqq c \| x\|_2^2$ for all $x \in M_+$, by density. Since R is a factor (cf. 3.7.5), by theorem 2.2 in [PiPo1] we get $[M:R] \leqq c^{-1}$. Q.E.D.

We now show that if we can choose the tunnel N_i so that $[M:R] < \infty$, with $R = \overline{\bigcup_i (N'_i \cap M)}$, then we can actually choose it so that $R = M$. We will prove this by contradiction. So we need to know what the failure of the generating property implies.

4.6 Lemma. *Let* $N \subset M$ *be so that there exists a choice of the tunnel* N_i, $i \geqq 1$, *with the property that* $R = \overline{\bigcup_i (N'_i \cap M)}$ *has finite index in* M. *Assume* $N \subset M$ *doesn't have the generating property. Then we have*
(1) $\forall \varepsilon > 0$, $\forall k_0 \geqq 1$, $\exists k_1 > k_0$, $\exists x \in M$, $\| x\|_2 = 1$, $\| x\| \leqq [M:R]^{1/2}$ *such that* $\| E_{u_1 R u_1^*}(x)\|_2 \leqq \varepsilon$, $\forall u_1 \in \mathscr{U}(N_{k_1})$.

Proof. Note first that the property (non 1) reads:

$$\text{(Non 1) ``}\exists \varepsilon_0 > 0,\ \exists k_0 \geqq 1 \text{ such that } \forall k > k_0,\ \forall x \in M \text{ with } \|x\|_2 = 1,$$

$$\|x\| \leqq [M:R]^{1/2},\ \exists u \in \mathcal{U}(N_k) \text{ such that } \|E_{uRu^*}(x)\|_2 > \varepsilon_0\text{''}\ .$$

Assume (Non 1), instead of (1), holds true. Let then $\{x_n\}_n \subset M$ be a sequence of elements satisfying $\|x_n\|_2 = 1$, $\|x_n\| \leqq [M:R]^{1/2}$, dense in the norm $\|\ \|_2$ in the set $\{x \in M \mid \|x\|_2 = 1,\ \|x\| \leqq [M:R]^{1/2}\}$. Let also $\{y_n\}_n \subset M_+$ be a sequence of elements dense in the norm $\|\ \|_2$ in M_+. We will find recursively $1 \leqq k_1 < k_2 < \ldots$ and a choice N_i^0, $i \geqq 1$, of the tunnel such that

(i) $\|E_{N_{k_n}^{0\prime} \cap M}(x_n)\|_2 \geqq \varepsilon_0$;
(ii) $\|E_{N_{k_n}^{0\prime} \cap M}(y_n)\|_2^2 \geqq {}_{1/n}[M:R]^{-1}\|y_n\|_2^2$;

Suppose $k_1 < k_2 < \ldots < k_{n-1}$ and $M \supset N \supset N_1^0 \supset \ldots \supset N_{k_{n-1}}^0$ have been chosen. Let $u_0 \in \mathcal{U}(N)$ be such that $u_0 N_i^0 u_0^* = N_i$ for $i = 1, \ldots, k_{n-1}$. Then

$$E_R(u_0 y_n u_0^*) \geqq [M:R]^{-1} u_0 y_n u_0^*\ ,$$

so that there exists $k_n^0 > k_{n-1}$, $k_n^0 > k_0$ such that:

$$E_{N_{k_n^0}^{\prime} \cap M}(u_0 y_n u_0^*) \geqq {}_{1/n}[M:R]^{-1} u_0 y_n u_0^*\ .$$

Thus we get

$$\begin{aligned} E_{u_0^* N_{k_n^0} u_0^{\prime} \cap M}(y_n) &= E_{u_0^*(N_{k_n^0}^{\prime} \cap M)u_0}(y_n) \\ &= u_0^* E_{N_{k_n^0}^{0\prime} \cap M}(u_0 y_n u_0^*) u_0 \geqq {}_{1/n}[M:R]^{-1} y_n\ . \end{aligned}$$

Moreover, by (Non 1) applied to $x = u_0 x_n u_0^*$, there exists $u_1 \in \mathcal{U}(N_{k_n^0})$ such that

$$\|E_{u_1 R u_1^*}(u_0 x_n u_0^*)\|_2 > \varepsilon_0\ .$$

But then, again by the definition of R, there exists $k_n > k_n^0$ with:

$$\|E_{u_1(N_{k_n}^{\prime} \cap M)u_1^*}(u_0 x_n u_0^*)\|_2 > \varepsilon_0\ .$$

Thus we also get:

$$\|E_{u_0^* u_1 N_{k_n} u_1^* u_0^{\prime} \cap M}(x_n)\|_2 = \|E_{u_0^* u_1(N_{k_n}^{\prime} \cap M)u_1^* u_0}(x_n)\|_2 > \varepsilon_0\ .$$

But $u_0^* u_1 N_i u_1^* u_0 = N_i^0$, for $1 \leqq i \leqq k_{n-1}$, so that if we put $N_j^0 = u_0^* u_1 N_j u_1^* u_0$ for $k_{n-1} < j \leqq k_n$ then both (i) and (ii) will be satisfied (also, since $u_0^* u_1 N_{k_n^0} u_1^* u_0 = u_0^* N_{k_n^0} u_0$). This ends the proof of the existence of N_j^0.

Let $R^0 = \overline{\bigcup_i (N_i^{0\prime} \cap M)}$. Then by (ii), $\|E_{R^0}(y)\|_2^2 \geqq [M:R]^{-1}\|y\|_2^2$, $y \in M_+$, so that $[M:R^0] \leqq [M:R]$. Since $R^0 \neq M$, $[M:R^0] \geqq 2$ so by [PiPo1] there exists $m \in M$ with $\|m\|_2 = 1$, $\|m\| \leqq [M:R]^{1/2}$ such that $E_{R^0}(m) = 0$. But (i) shows that $\|E_{R^0}(m)\|_2 \geqq \varepsilon_0$, a contradiction. Q.E.D.

4.7 Corollary. *Let ω be a free ultrafilter on* **N**. *With the notations and under the hypothesis of* 4.6, *there exist $x = (x_n)_n \in M^\omega$, $\|x\|_2 = 1$, $\|x\| \leqq [M:R]^{1/2}$, and there exists $k_1 < k_2 < \ldots$ such that $x \perp u R^\omega u^*$ for all unitary elements u in $\prod_{n \to \omega} N_{k_n}$.*

Proof. The construction of x_n follows by applying recursively 4.6. Q.E.D.

It turns out that if B_0, $B \subset M$ are von Neumann subalgebras of M then the span of uBu^* for u running in $\mathcal{U}(B_0)$ is almost a $B_0 - B_0$ bimodule. We will prove this technical result in the next section (see 5.1). Thus we get from 4.7:

4.8 Corollary. *With the notations and under the assumptions of* 4.6, 4.7, *there exists* $x \in M^\omega$, $\|x\|_2 = 1$ *such that* $x \perp (\prod_\omega N_{k_n}) R^\omega (\prod_\omega N_{k_n})$.

Proof. By 4.7 there exists $x \in M^\omega$, $\|x\|_2 = 1$, with

$$x \perp X \overset{\text{def}}{=} \text{span}\left\{ uR^\omega u^* \mid u \in \mathcal{U}\left(\prod_\omega N_{k_n} \right) \right\}.$$

Also, by 4.4 we have $\prod_\omega (N_{k_n} \vee (N'_{k_n} \cap M)) \subset X$. Indeed, if $y = (y_n)_n \in \prod_\omega (N_{k_n} \vee (N'_{k_n} \cap M))$ then by 4.4 for each n there is a $w_n \in \mathcal{U}(N_{k_n})$ and $l_n > k_n$ such that

$$y_n \in_{1/n} w_n (N'_{l_n} \cap M) w_n^* \subset w_n R w_n^* .$$

Thus $y \in wR^\omega w^* \subset X$, where $w = (w_n)_n$. Thus 5.1 applies and we get $x \perp (\prod_\omega N_{k_n}) R^\omega (\prod_\omega N_{k_n})$. Q.E.D.

We can now end up the proof of the generating result.

4.9 Theorem. *Let M be the hyperfinite II_1 factor and $N \subset M$ a subfactor of finite index. If N has finite depth (i.e. $\sup \dim \mathcal{Z}(M' \cap M_i) < \infty$ for the Jones tower M_i) then $N \subset M$ has the generating property (i.e. there exists a choice of the tunnel N_i, $i \geqq 1$, so that $\overline{\bigcup_i (N'_i \cap M)} = M$). Moreover, if $\tilde{M} = \overline{\bigcup_i M_i}$ then there is an anti-isomorphism of M onto $M' \cap \tilde{M}$ carrying N onto $M'_1 \cap \tilde{M}$.*

Proof. Assume $N \subset M$ doesn't have the generating property. By 4.5 and 4.8 there exists $x \in M^\omega$, $x \neq 0$, $x \perp \text{span}(\prod_\omega N_{k_n}) R^\omega (\prod_\omega N_{k_n})$, where N_i is a choice of the tunnel such that $R = \overline{\cup (N'_i \cap M)}$ satisfies $[M:R] < \infty$.

Since $E_R E_{N_i} = E_{R \cap N_i}$, $[R:N_i \cap R] \leqq [M:N_i]$ (cf. e.g. [Po2]). Moreover, since R contains the Jones projections in the tunnel, e_0, e_{-1}, e_{-2}, ..., it follows by [PiPo2] that R contains the Jones projection for $N_i \subset M$, i.e. a projection $f \in M$ with

$$E_{N_i}(f) = [M:N_i]^{-1} 1 .$$

Thus $f \in R$, $E_{N_i \cap R}(f) = E_{N_i}(f) = [M:N_i]^{-1} 1$, so that $[R:N_i \cap R] \geqq [M:N_i]$. Altogether $[M:N_i] = [R:N_i \cap R]$. But

$$[M:N_i][N_i:N_i \cap R] = [M:N_i \cap R] = [M:R][R:N_i \cap R]$$

(cf. [J1]) so that $[M:R] = [N_i:N_i \cap R]$. It follows that if $\{m_j^i\}_j$ is an orthonormal basis of N_i over $N_i \cap R$, then it is also an orthonormal basis of M over R, thus $\sum_j m_j^i R = M$. Put $m_j = (m_j^{k_n})_n \in M^\omega$. It follows that $\sum_i m_j R^\omega = M^\omega$. But $x \perp \sum_j m_j R^\omega$, a contradiction.

The rest follows by 3.2. Q.E.D.

Note that we also proved the following general result:

4.10 Theorem. *If $N \subset M$ are hyperfinite factors with finite index and if there is a choice of the tunnel N_k such that $R = \overline{\bigcup_k (N_k' \cap M)}$ is a factor and satisfies $E_R(x) \geqq cx$, for all $x \in M^+$ and some $c > 0$, then $N \subset M$ has the generating property.*

Proof. By the duality result 1.5 in [PiPo1] it follows that if $\overline{\cup (N_k' \cap M)}^w$ is a factor then $\overline{\cup (N_k' \cap N_1)}^w$ is also a factor, and more generally $\overline{\bigcup_k (N_k' \cap N_{2i+1})}^w$ are all factors for $i \geqq 0$. Moreover since $\overline{\cup (N_{2i+1}' \cap M)}^w = \overline{\cup (N_k' \cap M)}^w$, it follows that $N \subset M$ has the generating property iff $N_1 \subset M$ has the generating property. Altogether this shows that in the above statement we may assume $R_i = \overline{\cup (N_k' \cap M)}^w \cap N_i = \overline{\bigcup_i (N_k' \cap N_i)}^w$ are all factors. But then the proof of 4.9 shows that $N \subset M$ has the generating property. Q.E.D.

5. Proof of the linearization principle

We prove here the linearization principle that we used in the proof of Theorem 4.9. The essence of this linearization principle is the noncommutative Rohlin type theorem in [Po4].

5.1 Theorem. *Let M be a finite von Neumann algebra with a finite faithful trace τ. Let $B \subset M$ be a von Neumann subalgebra and $S \subset M$ a vector subspace. Then*

$$(\operatorname{span}\{uSu^* | u \in \mathcal{U}(B)\} + B \vee B' \cap M)^- = (\operatorname{span} BSB + B \vee B' \cap M)^- ,$$

the closures being taken in $L^2(M, \tau)$.

Proof. We clearly have the inclusion $\subset$. Let $\xi \in L^2(M, \tau)$ with $\langle uyu^*, \xi \rangle = 0$, $\langle b, \xi \rangle = 0$ for all $u \in \mathcal{U}(B)$, $y \in S$, $b \in B \vee B' \cap M$. Let $h_i \in B_h$, $i \leqq i \leqq n$. Pur $f: \mathbf{C}^n \to \mathbf{C}$,

$$f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \langle \exp(\lambda_1 h_1) \exp(\lambda_2 h_2) \ldots \exp(\lambda_n h_n) y \exp(-\lambda_n h_n)$$
$$\ldots \exp(-\lambda_1 h_1), \xi \rangle .$$

Then f is analytic on $\mathbf{C}^n$ and vanishes on $(i\mathbf{R})^n$. Thus f vanishes everywhere on $\mathbf{C}^n$. Since any invertible element in B is a product of two elements of the form $\exp(\lambda h)$, with $\lambda \in \mathbf{C}$, $h \in B_h$, we obtain that $\langle sys^{-1}, \xi \rangle = 0$, for any $y \in S$ and any invertible element s in B. Let $p \perp q$ be projections in B and $s \in B$ invertible. Denote $s_1(t) = tp + (1 - p)$, $s_2(t) = t(1 - q) + q$. Observe that $s_1(t)^{-1} = s_1(t^{-1})$, $s_2(t^{-1}) = s_2(t^{-1})$ and that

$$\lim_{t \to \infty} \left\| \frac{1}{t^2} s_2(t) s_1(t) sys^{-1} s_1(t)^{-1} s_2(t)^{-1} - psys^{-1}q \right\| = 0 .$$

It follows that $\langle psys^{-1}q, \xi \rangle = 0$. We now deduce that $\langle zyz^*, \xi \rangle = 0$ for all $z \in B$. It is clearly sufficient to prove this when $\operatorname{spec}(z^*z)$ is finite. So let $z = au$, where $a = \Sigma \alpha_i e_i$, with $\alpha_i \geqq 0$, $\Sigma e_i = 1$, $e_i \in \mathcal{P}(B)$, and $u \in \mathcal{U}(B)$. By [Po4] there is a refinement $f_j \in B$ of the e_i's so that

$$\left\| \sum_j f_j \xi f_j - e_{\sum_j f_j (B' \cap M)}(\xi) \right\|_2$$

is small (the result in [Po4] is stated for $B' \cap M \subset B$ but the general case used here has exactly the same proof). But since $\sum_j f_j(B' \cap M) \subset B \vee B' \cap M$ and $e_{B \vee B' \cap M}(\xi) = 0$, $e_{\Sigma f_j(B' \cap M)}(\xi) = 0$ as well. This shows that we can make $\| \sum_i f_j \xi f_j \|_2$ as small as we please, with $f_j \in \mathscr{P}(B)$ refining the e_i's.

Since $\langle f_i u y u^* f_j, \xi \rangle = 0$, for $i \neq j$, it follows that

$$\langle zyz^*, \xi \rangle = \langle auyu^*a, \xi \rangle = \left\langle \sum f_j auyu^* a f_j, \xi \right\rangle$$

$$= \left\langle auyu^*a, \sum_j f_j \xi f_j \right\rangle = \left\langle zyz^*, \sum_j f_j \xi f_j \right\rangle .$$

By the Cauchy-Schwartz inequality

$$|\langle zyz^*, \xi \rangle| \leqq \| zyz^* \|_2 \left\| \sum_j f_j \xi f_j \right\|_2$$

and for fixed z, y we can make this last term as small as we want. This shows that

$$\langle zyz^*, \xi \rangle = 0, z \in B, y \in S .$$

The standard polarization trick then implies: $\{z_1 y z_2, \xi\rangle = 0$, for all $z_1, z_2 \in B$, $y \in S$. Q.E.D.

6. Applications to the classification of subfactors

The generating type results reduce the classification (up to conjugacy) of the subfactors of the hyperfinite II_1 factor to the classification (up to trace preserving isomorphism) of the corresponding sequences of inclusions $\{N_i' \cap N \subset N_i' \cap M\}_i$ (or $\{M_1' \cap M_i \subset M' \cap M_i\}_i$). Moreover under the finite depth assumption all the information on such a sequence is contained in an initial commuting square:

$$\begin{array}{ccccc} N_{i_0}' & \cap & M & \subset & N_{i_0+1}' \cap M , \\ \cup & & & & \cup \\ N_{i_0}' & \cap & N & \subset & N_{i_0+1}' \cap N , \end{array}$$

where i_0, the inclusions, the dimensions and the traces are all determined by the standard matrices of $N \subset M$ and $N_1 \subset N$. To deduce this we return to the general setting of Section 2.

The next result was first noted by Jones and by Pimsner-Popa, independently ([JPP]).

6.1 Lemma. *Let*

$$\begin{array}{ccc} B_0 & \subset_A & B_1 \\ C \cup & & \cup D \\ B_0^1 & \subset_B & B_1^1 \end{array}$$

be a commuting square of finite dimensional algebras with the matrices A, B, C, D describing the corresponding inclusions. Assume $B_0 \subset B_1$, $B_0^1 \subset B_1^1$ satisfy the equivalent conditions of 1.2.2 *and $\|A\| = \|B\|$. If $B_0 \subset B_1 \subset^{e_2} B_2 \subset^{e_3} B_3 \subset \ldots$ is the*

iterated basic construction for $B_0 \subset B_1$, τ *is the* $\lambda = \|A\|^{-2}$*-Markov trace on* $\cup B_i$ *and if we denote* $B_{i+1}^1 = \text{span}\, B_i^1 e_{i+1} B_i^1$, $i \geqq 1$, *then* B_{i+1}^1 *has support one in* B_{i+1}, $B_0^1 \subset B_1^1 \subset^{e_2} B_2^1 \subset^{e_3} B_3^1 \subset \ldots$ *give the iterated basic construction for* $B_0^1 \subset B_1^1$, τ *is a* $\lambda = \|B\|^{-2}$*-Markov trace on* $\cup B_i^1$ *and we have the sequence of commuting squares*

$$\begin{array}{ccccccccc} B_0 & \subset_A & B_1 & \subset^{e_2}_{A'} & B_2 & \subset^{e_2}_{A} & B_3 & \subset & \ldots \\ \cup & & \cup & & \cup & & \cup & & \\ B_0^1 & \subset_B & B_1^1 & \subset^{e_2}_{B'} & B_2^1 & \subset^{e_3}_{B} & B_3^1 & \subset & \ldots \end{array}$$

Moreover, with the usual identifications of the centers of B_i, *respectively* B_i^1, *with the sets of indices of the matrices A, respectively B, the inclusion matrix for* $B_{2i}^1 \subset B_{2i}$ *is C and that of* $B_{2i+1}^1 \subset B_{2i+1}$ *is D,* $i \geqq 0$, *that is, in Wenzl's terminology* ([W]), *the sequence of commuting squares is periodic with period two.*

Proof. Let $B_0^1 \subset B_1^1 \subset^{\tilde{e}_2} \tilde{B}_2^1$ be the basic construction for $B_0^1 \subset B_1^1$ with a $\lambda = \|B\|^{-2}$-Markov trace τ_1 on $\tilde{B}_2^1$ satisfying $\tau_1(\tilde{e}_2 x) = \lambda\tau(x)$, $x \in B_1^1$, where τ is the Markov trace on B_2 (restricted to B_1^1). Define an isomorphism σ from $\tilde{B}_2^1$ onto B_2^1 by $\sigma(\Sigma x \tilde{e}_2 y) = \Sigma x e_2 y$. Since $\tau_1 \circ \sigma = \tau$ and $\tau_1(1) = 1$ it follows that $\tau_1(\sigma(1)) = 1$ so that $\sigma(1) = 1$ and thus B_2^1 has support 1 in B_2.

To show that the inclusion matrix for $B_2^1 \subset B_2$ is the same as for $B_0^1 \subset B_0$ note that $e_2 \in ((B_2^1)' \cap B_2)'$ and $e_2 \in ((B_0^1)' \cap B_0)'$. Note also that, since e_2 has support 1 in B_2^1, the map $(B_2^1)' \cap B_2 \ni z \mapsto ze_2$ is an isomorphism and that $(B_0^1)' \cap B_0 \ni z \mapsto ze_2$ is an isomorphism as well. Since

$$e_2((B_2^1)' \cap B_2)e_2 = (e_2 B_2^1 e_2)' \cap e_2 B_2 e_2 = (B_0^1 e_2)' \cap B_0 e_2 = ((B_0^1)' \cap B_0)e_2$$

it follows that there is a unique isomorphism $\varphi: (B_2^1)' \cap B_2 \to (B_0^1)' \cap B_0$ determined by $\varphi(z)e_2 = ze_2$. But this isomorphism, when restricted to $\mathscr{Z}(B_2)$, $\mathscr{Z}(B_2^1)$, gives the usual identifications between $\mathscr{Z}(B_2)$ and $\mathscr{Z}(B_0)$, respectively $\mathscr{Z}(B_2^1)$ and $\mathscr{Z}(B_0^1)$. Moreover the dimensions of $((B_2^1)' \cap B_2)q'q''$ for minimal projections q' in $\mathscr{Z}(B_2)$, q'' in $\mathscr{Z}(B_2^1)$, give the multiplicities in the inclusion matrix of $B_2^1 \subset B_2$, which will therefore coincide with that of $B_0^1 \subset B_0$. Q.E.D.

The next result will not be used here but it may be useful for constructing examples of subfactors. Related results on periodic sequences of commuting squares can be found in [W].

6.2 Corollary. *Let*

$$\begin{array}{ccccc} & & B_0 & \subset_A & B_1 & & \\ C & \cup & & & \cup & D \\ & & B_0^1 & \subset_B & B_1^1 & & \end{array}$$

be like in 6.1 *and let*

$$\begin{array}{ccccccccc} B_0 & \subset_A & B_1 & \subset^{e_2}_{A'} & B_2 & \subset^{e_3}_{A} & B_3 & \subset & \ldots \\ \cup & & \cup & & \cup & & \cup & & \\ B_0^1 & \subset_B & B_1^1 & \subset^{e_2}_{B'} & B_2^1 & \subset^{e_3}_{B} & B_3^1 & \subset & \ldots \end{array}$$

be the corresponding iterated sequence of commuting squares. Let $M = \overline{\bigcup_n B_n}^w$, $N = \overline{\bigcup_n B_n^1}^w$. *Then*

1°. *M is a factor iff A is irreducible*

2°. *N is a factor iff B is irreducible*

3°. *If N, M are factors then*

(a) $[M:N] = \|C\|^2 = \|D\|^2$

(b) $H(M|N) = \ln[M:N]$ *so that* $E_{N'\cap M}(e_0) \in \mathbf{C}$ *for the Jones projection* $e_0 \in M$.

Proof. 1°, 2° follow by Section 2 and 3° (a) is a particular case of 1.5 in [W]. Then 3°, (b) follows by [PiPo1] and, again, by [W]. Q.E.D.

6.3 Lemma. *If* $N \subset M$ *has finite depth then* $N_i \subset N_{i-1}$ *and* $M_{i-1} \subset M_i$ *have finite depth for all* $i \geqq 1$. *Moreover the standard matrices for* $N_{2i+1} \subset N_{2i}$, $M_{2i} \subset M_{2i+1}$ *coincide with that of* $N_1 \subset N$ *and the standard matrices of* $N_{2i} \subset N_{2i-1}$, $M_{2i-1} \subset M_{2i}$ *coincide with that of* $N \subset M$.

Proof. If C_i is the matrix of the inclusion $N_i' \cap N \subset N_i' \cap M$ then by 2.9 in [PiPo3] we have $\|C_i\|^2 \leqq [M:N]$. Since $\sup \dim \mathscr{Z}(N_i' \cap M) < \infty$ it follows that $\sup \dim \mathscr{Z}(N_i' \cap N) < \infty$. Thus $N_1 \subset N$ has finite depth. The rest follows by 1.8 in [PiPo1]. Q.E.D.

The previous lemma and 6.1 show that in the finite depth case the sequence of commuting squares

$$\begin{array}{ccccccc} M' \cap M_{2i_0} & \subset & M' \cap M_{2i_0+1} & \subset & M' \cap M_{2i_0+2} & \subset & \dots \\ \cup & & \cup & & \cup & & \\ M_1' \cap M_{2i_0} & \subset & M_1' \cap M_{2i_0+1} & \subset & M_1' \cap M_{2i_0+2} & \subset & \dots \end{array}$$

for i_0 large enough comes by iterating the basic construction for the first commuting square, like in 6.1. The next result clarifies what are the inclusion matrices.

6.4 Lemma. *Assume* $N \subset M$ *has finite depth and let* $A = (a_{kl})_{\substack{k \in K \\ l \in L}}$, $k_0 \in K$, $A^1 = (a_{k^1 l^1})_{\substack{k^1 \in K^1 \\ l^1 \in L^1}}$, $k_0^1 \in K^1$, *be the standard matrices of* $N \subset M$ *respectively* $M \subset M_1$. *Let* M_n *be the Jones' tower of factors and* $i_0 \geqq \operatorname{card} K$, $\operatorname{card} L^1 + 1$. *Then the commuting square*

$$\begin{array}{ccc} M' \cap M_{2i_0} & \subset_A & M' \cap M_{2i_0+1} \\ \cup & & \cup \\ M_1' \cap M_{2i_0} & \subset_{A^1} & M_1' \cap M_{2i_0+1} \end{array}$$

has the corresponding inclusion matrices and satisfies the conditions of 6.1. *Moreover there are trace preserving isomorphisms between the pairs* $M_1' \cap M_{2i_0} \subset M' \cap M_{2i_0}$ *and* $M' \cap M_{2i_0-1} \subset M' \cap M_{2i_0}$ *and respectively* $M_1' \cap M_{2i_0+1} \subset M' \cap M_{2i_0+1}$ *and* $M_1' \cap M_{2i_0+1} \subset M_1' \cap M_{2i_0+2}$. *Via these identifications the inclusion matrices for* $M_1' \cap M_{2i_0} \subset M' \cap M_{2i_0}$ *and* $M_1' \cap M_{2i_0+1} \subset M' \cap M_{2i_0+1}$, *in the commuting square, are* A^t *respectively* A^1, *in particular* $\operatorname{card} L = \operatorname{card} L^1$.

Proof. The first part is clear by 6.3. The last part follows by observing that if J_{i_0}, J_{i_0+1} are the canonical conjugations in $L^2(M_{i_0}, \tau)$ respectively $L^2(M_{i_0+1}, \tau)$ then

$J_{i_0}(M' \cap M_{2i_0})J_{i_0} = M' \cap M_{2i_0}$, $J_{i_0}(M'_1 \cap M_{2i_0})J_{i_0} = M' \cap M_{2i_0-1}$ and $J_{i_0+1}(M'_1 \cap M_{2i_0+1})J_{i_0+1} = M'_1 \cap M_{2i_0+1}$, $J_{i_0+1}(M' \cap M_{2i_0+1})J_{i_0+1} = M'_1 \cap M_{2i_0+2}$. But by 6.3, 3.2, the corresponding applications $J_{i_0} \cdot J_{i_0}$, $J_{i_0+1} \cdot J_{i_0+1}$ are trace preserving. Since the algebras are finite dimensional it follows that there exist the required trace preserving isomorphisms as well. Q.E.D.

We mention that the use of the reflections $J_i \cdot J_i$ as antiautomorphisms of $M' \cap M_{2i}$ was discovered by M. Pimsner and the author and, independently, by Ocneanu ([Oc]). Moreover in ([Oc]) the reflections and the commuting square conditions are suitably used to give a complete intrinsic description of the sequence $\{M'_1 \cap M_i \subset M' \cap M_i\}$.

6.5 Definition. If $N \subset M$ has finite depth and $A = (a_{kl})_{\substack{k \in K \\ l \in L}}$, is its standard matrix and if $i_0 = \max\{\text{card } K, \text{card } L + 1\}$ then

$$\begin{array}{ccc} M' \cap M_{2i_0} & \subset & M' \cap M_{2i_0+1} \\ \cup & & \cup \\ M'_1 \cap M_{2i_0} & \subset & M'_1 \cap M_{2i_0+1} \end{array}$$

is called the *canonical commuting square* associated with $N \subset M$.

We say that two commuting squares

$$\begin{array}{ccc} B^0_{0i} & \subset & B^0_{1i} \\ \cup & & \cup \\ B^1_{0i} & \subset & B^1_{1i} \end{array}$$

$i = 1, 2$, are isomorphic iff there is a trace preserving isomorphism of B^0_{11} onto B^0_{12} carrying B^i_{j1} onto B^i_{j2}, $i, j = 0, 1$.

We can now derive the classification result for subfactors, in terms of classification of their canonical commuting squares.

6.6 Theorem. *Let $N \subset M$, $N^0 \subset M^0$ be pairs of hyperfinite type II_1 factors with finite depth. Then $N \subset M$ and $N^0 \subset M^0$ are isomorphic iff their associated canonical commuting squares are isomorphic. Thus, the subfactors of finite depth of the hyperfinite type II_1 factor are completely classified by their canonical commuting squares.*

Proof. If the commuting squares are isomorphic then their corresponding iterated sequences will be isomorphic by 6.1, i.e. there will be a trace preserving $*$-isomorphism of $\cup(M' \cap M_n)$ onto $\bigcup_n(M^{0\prime} \cap M^0_n)$ carrying $\overline{M' \cap M_n}$ onto $\overline{M^{0\prime} \cap M^0_n}$, $\overline{M'_1 \cap M_n}$ onto $\overline{M^{0\prime}_1 \cap M^0_n}$. Taking $M_0 = \overline{\cup(M' \cap M_n)^w}$, $N_0 = \overline{\cup(M'_1 \cap M_n)^w}$, $M^0_0 = \overline{\bigcup_n(M^{0\prime} \cap M^0_n)^w}$, $N^0_0 = \overline{\bigcup_n(M^{0\prime}_1 \cap M^0_n)^w}$, it follows that $N_0 \subset M_0$ is isomorphic to $N^0_0 \subset M^0_0$. By Theorem 4.9 it follows that $N \subset M$ is isomorphic to $N^0 \subset M^0$. The converse implication is trivial. Q.E.D.

As the previous theorem shows, classifying the subfactors of finite depth amounts to the investigation of the corresponding commuting squares. By 6.4 the algebras, the traces and the horizontal inclusions involved are completely deter-

mined by the standard (pointed!) matrices of $N \subset M$, $M \subset M_1$. Moreover the vertical inclusions are determined by the same matrices up to some possible permutations of the sets of indices. These matrices and the permutations must satisfy some very strong conditions of matching the traces and the dimensions: for instance by 6.4 the Perron Frobenius eigenvectors of A^tA and $A^{1t}A^1$ must coincide (via the identification of L, L^1 resulting from 6.4). Also the dimensions should satisfy $\tilde{A}(A^1)^i(A^1A^{1t})^{i-1}v_0^1 = (AA^t)^iv_0$, where $v_0 = (\delta_{k_0k})_k$, $v_0^1 = (\delta_{k_0^1k^1})_{k^1}$ and $\tilde{A} = (\tilde{a}_{kl^1})_{\substack{k\in K\\ l^1\in L^1}}$ with $\tilde{a}_{kl^1} = a_{kl}$, $l^1 \to l$ being the identification of L with L^1 given by 6.4.

In many situations these conditions impose $A^1 = A$. This can be easily seen for $\|A\|^2 = [M:N] < 4$, where one has to analyse only the matrices coming from the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 and in fact even for $\|A\|^2 < 2 + \sqrt{5}$. Then direct simple computations can be used to deduce that if

$$\begin{array}{ccc} B_0 & \subset_A & B_1 \\ {\scriptstyle A^t}\cup & & \cup\ {\scriptstyle A} \\ B_0^0 & \subset & B_1^0 \\ & {\scriptstyle A^t} & \end{array}$$

is a commuting square with $A = A_n$, D_n, E_6, E_7, E_8 then there will be no small perturbations of B_1^0 to still satisfy the commuting square condition with B_0, B_0^0, unless it is of the form $vwB_1^0w^*v^*$ for some unitary elements $w \in B_0$, $v \in B_0^1 \cap B_1$, with $wB_0^0w^* = B_0^0$. Thus the number of commuting squares in 6.5 with given matrices is bounded. But if $[M:N] < 4$ then $N \subset M$ automatically has finite depth and $N' \cap M = \mathbf{C}$ (as first noted by Jones). Indeed if $[M:N] < 4$ and $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ is the Jones' tower then $e_1 \vee e_2 \vee \ldots \vee e_n = 1$ (cf. [J1]) so $\mathrm{Alg}\{e_1, \ldots, e_{n-1}\}e_n\ \mathrm{Alg}\{e_1, \ldots, e_{n-1}\}$ has support 1. Thus $N' \cap M_{n-1}e_nN' \cap M_{n-1}$ has support 1 so $N \subset M$ has finite depth by Section 3. Thus Theorem 6.6 completely classifies the subfactors of index less than 4 in terms of their associated commuting squares, coming from their higher relative commutants picture, which in turn, by the above discussion, are finite in number. In fact, in case $\|A\|^2 < 4$, the above observation on the combinatorial problem of estimating the number of canonical commuting squares is superceded by the complete list of such objects corresponding to indices less than 4 announced in [Oc]. We use in the next corollary only the finiteness of the number of canonical commuting squares of small indices resulting from the above remarks, without however giving further details on the proof.

6.7 Corollary. *Up to conjugacy, there are only finitely many subfactors N of index $4\cos^2\frac{\pi}{n}$ in the hyperfinite II_1 factor M, for each $n \geqq 3$, and all have finite depth. Moreover if the subfactor N has only the Jones projections $e_1, \ldots, e_n$ in its relative commutant in M_n then $N \subset M$ coincides with the Jones' subfactor of index $4\cos^2\frac{\pi}{n}$.*

Note that the last part of the corollary is just a simple consequence of 4.9 and of Jones' result ([J1]) that if $[M:N] = 4\cos^2\frac{\pi}{n}$ and $N' \cap M_n = \text{Alg}\{e_1, \ldots, e_n\}$ then $N' \cap M_k = \text{Alg}\{e_1, \ldots, e_k\}$ for all k. Thus $M_1' \cap \tilde{M} \subset M' \cap \tilde{M}$ coincides with the Jones' pair of factors of index $4\cos^2\frac{\pi}{n}$ which is antiisomorphic to itself and thus it is isomorphic to $N \subset M$ by 4.9.

We mention that one can prove under very general assumptions that the commuting square problem has only finitely many solutions (up to isomorphism), when the dimensions of the algebras involved are fixed.

As concerning the index > 4, the case $< 2 + \sqrt{5}$ is still tractable, since the matrices of such small norm can actually be computed. One can show for instance that only finitely many of them may produce commuting squares like in 6.4. A first result of this type was proved by M. Pimsner and the author (1983 unpublished) for finite matrices tending to the infinite matrix:

In fact to show this it is sufficient to consider the infinite matrix itself describing the matricial inclusion of type II_1 algebras with atomic centers like in [PiPo3], where the problem becomes quite easy. The same arguments can be used to settle the case of the other infinite matrices (and thus accumulation points) of square norm less than $2 + \sqrt{5}$.

This shows that there can be only finitely many subfactors of finite depth and index between 4 and $2 + \sqrt{5}$ in the hyperfinite II_1 factor. Note however that although by [PiPo1] subfactors of index $< 2 + \sqrt{5}$ satisfying $E_{N' \cap M}(e_0) \in \mathbf{C}$ automatically satisfy $N' \cap M = \mathbf{C}$, the finite depth condition doesn't follow automatically.

References

[C] Connes, A.: Classification of injective factors. Ann. Math. **104**, 73–115 (1976)

[J1] Jones, V.F.R.: Index for subfactors. Invent. Math. **72**, 1–25 (1983)

[J2] Jones, V.F.R.: Subfactors and related topics, Operator Algebras and Applications, Vol. 2, London Math. Soc. Lect. Note Series **136**, 103–118 (1988)

[J3] Jones, V.F.R.: Actions of finite groups on the hyperfinite II_1 factor. Mem. Am. Math. Soc. **28**, No. 237 (1980)

[JPP] Jones, V., Pimsner, M., Popa, S.: Private correspondence, 1982–1984

[GHJ] Goodman, F., de la Harpe, P., Jones, V.F.R.: Coxeter graphs and towers of algebras. MSRI Publications 14, Springer Verlag, 1989

[MvN] Murray, F., von Neumann, J.: Rings of operators IV. Ann. Math. **44**, 716–808 (1943)

[Oc] Ocneanu, A.: Quantized groups string algebras and Galois theory for algebras. Operator Algebras and Applications, Vol. 2, London Math. Soc. Lect. Note Series **136**, 119–172 (1988)

[PiPo1] Pimsner, M., Popa, S.: Entropy and index for subfactors. Ann. Sci. Ec. Norm. Super., IV. Ser. **19**, 57–106 (1986)

[PiPo2] Pimsner, M., Popa, S.: Iterating the basic construction. Trans. Am. Math. Soc. **310**, 127–134 (1988)

[PiPo3] Pimsner, M., Popa, S.: Finite dimensional approximation of pairs of algebras and obstructions for the index. Preprint 1988

[Po1] Popa, S.: Maximal injective subalgebras in factors associated with free groups. Adv. Math. **50**, 27–48 (1983)

[Po2] Popa, S.: Relative dimension, towers of projections and commuting squares of subfactors. Pac. J. Math. **137**, 95–122 (1989)

[Po3] Popa, S.: Matrices entiéres et algébres envelopantes associées aux sousfacteurs. Preprint 1990

[Po4] Popa, S.: On a problem of R.V. Kadison on maximal abelian $*$-subalgebras in factors. Invent. Math. **65**, 269–281 (1981)

[Wa] Wassermann, A.: Coactions and Yang-Baxter equations for ergodic actions and subfactors. Operator Algebras and Applications, Vol. 2, London Math. Soc. Lect. Note Series **136**, 203–236 (1988)

[W] Wenzl, H.: Hecke algebras of type A_n and subfactors. Invent. Math. **92**, 349–383 (1988)

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Invent. math. 101, 45–56 (1990)

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Cauchy-Riemann functions on manifolds of higher codimension in complex space

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§1. Introduction and main results

A smooth manifold M is called *CR* if there exists a subbundle $\mathscr{V}$ of $\mathbf{C}TM$, the complexified tangent bundle of M, such that

$$\mathscr{V} \cap \bar{\mathscr{V}} = 0 \quad \text{and} \quad [\mathscr{V}, \mathscr{V}] \subset \mathscr{V} . \tag{1.1}$$

A function or distribution on M is called *CR* if it is annihilated by all the sections of $\mathscr{V}$. If $\dim_{\mathbf{R}} M = 2n + l$ and $\dim_{\mathbf{C}} \mathscr{V} = n$ then we say that M is of *CR dimension* n and *CR codimension* l.

If $(M, \mathscr{V})$ is a CR manifold then N is called a *CR submanifold* of M (of the same CR dimension) if N is a submanifold of M satisfying

$$\mathscr{V}|_N \subset \mathbf{C}TN . \tag{1.2}$$

Note that it follows from (1.1) and (1.2) that

$$2\dim_{\mathbf{C}} \mathscr{V} \leqq \dim_{\mathbf{R}} N \leqq \dim_{\mathbf{R}} M , \tag{1.3}$$

and M and N have the same CR dimension. In what follows it will be understood that by a CR submanifold of M we will mean that (1.2) is satisfied.

We introduce the following definition.

Definition. If M is a CR manifold then M is *minimal at* $m_0 \in M$ if there is no CR submanifold N containing m_0 with $\dim_{\mathbf{R}} N < \dim_{\mathbf{R}} M$.

The importance of this notion of minimality was introduced by Tumanov [14], who showed that if M is a generic embedded submanifold of $\mathbf{C}^{n+l}$ and $m_0 \in M$ then if M is minimal at m_0 every CR function on M, defined in a neighborhood of m_0, is the boundary value of a holomorphic function in an open wedge of $\mathbf{C}^{n+l}$ of edge M.

It is easy to see that if M is of finite type in the sense of Bloom-Graham [6] then M is minimal at m_0. Indeed finite type means that the Lie algebra of the sections of

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$\mathscr{V}$ and their conjugates span the complexified tangent space at m_0. If these are also tangent to N then the germs of N and M are the same at m_0. If $(M, \mathscr{V})$ is real analytic then minimal implies finite type; indeed, if M is not of finite type, then the Nagano leaf [9] passing through m_0 would be a proper real analytic CR submanifold, contradicting minimality. If M is only smooth, it could be minimal without being of finite type.

The main result of this paper is to prove that Tumanov's minimality condition is also necessary for holomorphic extendability. Recall that a CR manifold M of CR dimension n and CR codimension l is *locally embeddable* if in a neighborhood of every point there exist $n + l$ smooth CR functions with linearly independent differentials.

Theorem 1. *Let M be a locally embeddable CR manifold which is not minimal at $m_0 \in M$ and N a CR submanifold of M containing m_0, with $\dim_{\mathbf{R}} N < \dim_{\mathbf{R}} M$. Then there exists a CR distribution T defined in a neighborhood U of m_0 with* $\operatorname{supp} T = N \cap U$.

A slightly more general version of Theorem 1 is given in §6 as Theorem 4.

A submanifold M of $\mathbf{C}^{n+l}$ of dimension $2n + l$ is called *generic* if it is locally defined, near $m_0 \in M$, by $\rho_j = 0$, $j = 1, \ldots, l$, where the ρ_j are smooth, real functions such that their complex differentials $\partial\rho_j$ are linearly independent. Such a manifold M equipped with $\mathscr{V}$, the induced tangential Cauchy-Riemann bundle of $\mathbf{C}^{n+l}$, is a CR manifold of CR dimension n and CR codimension l.

Recall that a wedge of edge M is an open set of $\mathbf{C}^{n+l}$ of the form

$$\mathscr{W}(\mathcal{O}, \Gamma) = \{Z \in \mathcal{O},\ \rho(Z) \in \Gamma\}\ ,$$

where $\mathcal{O}$ is a sufficiently small open neighborhood of m_0 in $\mathbf{C}^{n+l}$, Γ an open cone of $\mathbf{R}^l$, and $\rho = (\rho_1, \ldots, \rho_l)$ where the ρ_j are the defining functions of M near m_0 as above. We can now state our nonextendability result.

Theorem 2. *If M is an embedded generic CR manifold which is not minimal at m_0 then there is a smooth CR function defined in a neighborhood of m_0 which does not extend holomorphically to any wedge with edge U, where U is a neighborhood of m_0 in M.*

Combining Theorem 2 with the sufficient condition of Tumanov we obtain the following.

Corollary 1. *If M is an embedded generic CR manifold and $m_0 \in M$, then every germ of a CR function at m_0 extends holomorphically to a wedge of edge M if and only if M is minimal at m_0.*

The following result shows uniqueness of a CR submanifold of minimal dimension and gives two intrinsic characterizations of such a submanifold.

Theorem 3. *If M is a generic CR manifold and $m_0 \in M$, there is a unique germ N_0 of a CR submanifold contained in M, $m_0 \in N_0$, of minimal dimension. Also N_0 may be described as follows.*

(i) *For every sufficiently small neighborhood U of m_0 in M, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ consists of all points $m \in U'$ which can be reached from m_0 by a finite sequence of integral curves contained in U of sections of* $\operatorname{Re}\mathscr{V}$.

(ii) *For every sufficiently small neighborhood U of m_0 in M, there is a neighborhood $U' \subset U$ such that $N_0 \cap U'$ is the union of sets of the form $Z(bD)$, where D is the unit disc in $\mathbf{C}$ and Z: $\bar{D} \to \mathbf{C}^{n+l}$ is continuous, holomorphic in D and satisfying $m_0 \in Z(bD) \subset U \cap M$.*

We note that the uniqueness of N_0 and its characterization given by (i) in Theorem 3 hold even in the case of a nonembeddable CR manifold.

The study of extendability of CR functions began with the celebrated paper of Hans Lewy [8] in the '50's in which he showed extendability to one side from a strictly pseudoconvex hypersurface. Other sufficient conditions were subsequently obtained by a number of mathematicians. Recently, a necessary and sufficient condition for every CR function to extend to at least one side of a hypersurface of class C^2 was obtained by Trépreau [12]. The authors obtained some sufficient conditions [2] for extendability in the higher codimension case; these results have been generalized by Tumanov, as explained above. They also obtained in the real analytic case (loc. cit.) necessary conditions, i.e. that M must be of finite type, in order that extendability to a wedge hold. Since Tumanov proved that finite type is also sufficient for extendability, the case of real analytic generic manifolds was completed by Tumanov's work. Our Theorem 2 above now settles the smooth case.

Singular solutions for real analytic vector fields supported on submanifolds were constructed by Zachmanoglou [15] in the study of analytic hypoellipticity and propagation of zeroes. Our solutions in Theorem 1 have the same general form as his, but in our case the vector fields are not real analytic.

We would like to thank Jean-Marie Trépreau for several interesting conversations and, in particular, for suggesting to us the formulation of Theorem 3, and the use of Sussmann's work [11] in its proof.

§2. Local coordinates

We assume here that M is an embedded generic CR manifold which is not minimal and N a CR submanifold of M passing through m_0. We first introduce local coordinates around m_0 which will be used in the proof of Theorem 1. Assume that

$$\dim_{\mathbf{R}} M = 2n + l, \quad \dim_{\mathbf{R}} N = 2n + l_1 \text{ and } \dim_{\mathbf{C}} \mathcal{V} = n\,, \tag{2.1}$$

with $0 \leqq l_1 < l$.

We may choose a local embedding so that M is parametrized in $\mathbf{C}^{n+l}$, where the coordinates are denoted by $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_l)$, by

$$\operatorname{Im} w_j = \phi_j(z, \bar{z}, s), \quad 1 \leqq j \leqq l, \quad s = \operatorname{Re} w,\ \phi(0) = d\phi(0) = 0\,. \tag{2.2}$$

A basis of CR vector fields L_j, $j = 1, \ldots, n$, can then be written in the form

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{1 \leqq k \leqq l} \alpha_{jk}(z, \bar{z}, s) \frac{\partial}{\partial s_k}\,, \tag{2.3}$$

with $\alpha_{jk}(0) = 0$.

Assume that N is given by $\rho_j(z, \bar{z}, s) = 0$, $1 \leqq j \leqq l_2$, where $l_2 = l - l_1$ and the differentials $d\rho_j$ are linearly independent. Since the L_j are tangent to N, we must

have $L_j\rho_k = 0$ on N, and $\bar{L}_j\rho_k = 0$ on N, since the ρ_j are real valued. Hence, by (2.3), $\dfrac{\partial\rho_j}{\partial x_k}(0) = \dfrac{\partial\rho_j}{\partial y_k}(0) = 0$. We conclude that $rank\left(\dfrac{\partial\rho_j}{\partial s_k}(0)\right)_{\substack{1\leqq j\leqq l_2\\ 1\leqq k\leqq l}} = l_2$. After a linear change of variables and using the implicit function theorem, we can assume that N is given as a subset of M by

$$s_{l_1+j} = \psi_j(z, \bar{z}, s_1, \ldots, s_{l_1}), \quad 1\leqq j\leqq l_2, \quad \psi_j(0) = d\psi_j(0) = 0\,. \tag{2.4}$$

Put $t_j = s_{l_1+j} - \psi_j(z, \bar{z}, s)$, $j = 1, \ldots, l_2$, and take for coordinates on M,

$$(x, y, s, t) = (x, y, s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2})\,.$$

In these coordinates the L_j become

$$L_j = \frac{\partial}{\partial\bar{z}_j} + \sum_{k=1}^{l_1}\beta_{jk}\frac{\partial}{\partial s_k} + \sum_{\substack{1\leqq k\leqq l_2\\ 1\leqq r\leqq l_2}}\mu_{jkr}\,t_k\frac{\partial}{\partial t_r}\,, \tag{2.5}$$

where β_{jk} and μ_{jkr} are functions of (x, y, s, t). We introduce the vector fields L_j^0, obtained from L_j by setting $t = 0$:

$$L_j^0 = \frac{\partial}{\partial\bar{z}_j} + \sum_{k=1}^{l_1}\beta_{jk}(x, y, s, 0)\frac{\partial}{\partial s_k}\,. \tag{2.6}$$

The L_j^0 form a basis of the sections of the CR bundle restricted to N. We set

$$\begin{aligned} f_j &= s_j + i\phi_j(z, \bar{z}, s, t + \psi(x, y, s)), \quad 1\leqq j\leqq l_1\,,\\ g_j &= t_j + \psi_j(z, \bar{z}, s) + i\phi_{j+l_1}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)), \quad 1\leqq j\leqq l_2\,, \end{aligned} \tag{2.7}$$

where the ϕ_j are given by (2.2) and ψ_j by (2.4). Note that we have $L_jf_k = L_jg_k = 0$.

For a smooth function $h(x, y, s, t)$ we write

$$h(x, y, s, t) = h^0(x, y, s) + \sum_{k=1}^{l_2}h^k(x, y, s)t_k + O(|t|^2)\,.$$

Since $L_jf_k = 0$ we conclude that $L_j^0f_k^0 = 0$, $1\leqq j\leqq n$, $1\leqq k\leqq l_1$. As in [3] we define l_1 vector fields R_j of the form $R_j = \sum_{k=1}^{l_1}a_{jk}(x, y, s)\dfrac{\partial}{\partial s_k}$ satisfying the following relations:

$$[L_k^0, R_j] = 0, \qquad [R_j, R_p] = 0, \qquad R_jf_p^0 = \varepsilon_{jp}\,, \tag{2.8}$$

where ε_{jp} is the Kronecker symbol. It follows from (2.7) and (2.8) that we have

$$L_j^0 = \frac{\partial}{\partial\bar{z}_j} - \sum_{k=1}^{l_1}f_{k\bar{z}_j}^0R_k\,. \tag{2.9}$$

Since the R_k are linear independent we obtain, for $j = 1, \ldots, n$,

$$L_j = L_j^0 + \sum_{\substack{1\leqq k\leqq l_2\\ 1\leqq p\leqq l_1}}\lambda_{jkp}t_kR_p + \sum_{\substack{1\leqq k\leqq l_2\\ 1\leqq r\leqq l_2}}\mu_{jkr}t_k\frac{\partial}{\partial t_r}\,. \tag{2.10}$$

§3. Construction of singular solutions. Proof of Theorem 1

We shall use the coordinates (x, y, s, t) introduced in §2 and take a basis for the sections of $\mathscr{V}$ of the form (2.10). We shall construct a distribution solution T of the system of equations $L_j T = 0$ of the form $T = V(z, \bar{z}, s)\ \delta(t)$, where $\delta(t) = \delta(t_1) \otimes \ldots \otimes \delta(t_{l_2})$ is the dirac measure at the origin in $\mathbf{R}^{l_2}$ and V is a smooth function nonvanishing at 0.

Using the relation

$$t_k \frac{\partial}{\partial t_r} \delta(t) = -\varepsilon_{kr} \delta(t)$$

we conclude that V must satisfy the equations

$$L_j^0 V - \sum_{1 \leqq r \leqq l_2} \mu_{jrr}^0(z, \bar{z}, s) V = 0, \quad 1 \leqq j \leqq n\,. \tag{3.1}$$

Since V is nonvanishing, equation (3.1) is equivalent to

$$L_j^0(\operatorname{Log} V) - \sum_{1 \leqq r \leqq l_2} \mu_{jrr}^0(z, \bar{z}, s) = 0\,. \tag{3.2}$$

Therefore, it suffices to show that $\sum_{1 \leqq r \leqq l_2} \mu_{jrr}^0(z, \bar{z}, s)$ is in the range of L_j^0. We begin with the calculation of the coefficients μ_{jrr}^0. It follows from (2.10), that if $L_j h = 0$ then

$$L_j h = \left(L_j^0 + \sum_{\substack{1 \leqq k \leqq l_2 \\ 1 \leqq p \leqq l_1}} \lambda_{jkp}^0 t_k R_p + \sum_{\substack{1 \leqq k \leqq l_2 \\ 1 \leqq r \leqq l_2}} \mu_{jkr}^0 t_k \frac{\partial}{\partial t_r} \right) \left(h^0 + \sum_{1 \leqq k \leqq l_2} h^k t_k \right) + O(t^2)\,, \tag{3.3}$$

where we have used the notation $h = h^0 + \sum_{1 \leqq k \leqq l_2} h^k t_k + O(t^2)$ as in §2.

Setting the coefficient of t_k equal to 0 in (3.3) we obtain

$$L_j^0 h^k + \sum_{1 \leqq p \leqq l_1} \lambda_{jkp}^0 R_p h^0 + \sum_{1 \leqq r \leqq l_2} \mu_{jkr}^0 h^r = 0\,. \tag{3.4}$$

Taking for h the functions f_q, $1 \leqq q \leqq l_1$, defined in (2.7), and using the relations $R_p f_k^0 = \varepsilon_{pk}$ in (2.8), we obtain,

$$\lambda_{jkp}^0 = -L_j^0 f_p^k - \sum_{1 \leqq r \leqq l_2} \mu_{jkr}^0 f_p^r,\ 1 \leqq j \leqq n, \quad 1 \leqq k \leqq l_2, \quad 1 \leqq p \leqq l_1\,. \tag{3.5}$$

In (3.4) we now replace h by g_q, $1 \leqq q \leqq l_2$, defined in (2.7) and λ_{jkp}^0 by its expression given in (3.5). We obtain

$$L_j^0 g_q^k + \sum_{1 \leqq p \leqq l_1} \left(-L_j^0 f_p^k - \sum_{1 \leqq r \leqq l_2} \mu_{jkr}^0 f_p^r \right) R_p g_q^0 + \sum_{1 \leqq r \leqq l_2} \mu_{jkr}^0 g_q^r = 0\,. \tag{3.6}$$

Rewriting (3.6) by collecting the coefficients of μ_{jkr}^0 we have

$$\sum_{1 \leqq r \leqq l_2} \mu_{jkr}^0 \left(g_q^r - \sum_{1 \leqq p \leqq l_1} f_p^r R_p g_q^0 \right) = -L_j^0 g_q^k + \sum_{1 \leqq p \leqq l_1} (L_j^0 f_p^k) R_p g_q^0\,. \tag{3.7}$$

Let $\mathscr{A}$ be the matrix $(g_q^r - \sum_{1 \leqq p \leqq l_1} f_p^r R_p g_q^0)_{1 \leqq r, q \leqq l_2}$ and $D(z, \bar{z}, s) = \det \mathscr{A}$. By Cramer's rule applied to (3.7) we obtain

$$\mu_{jkr}^0 = \frac{D_{jkr}}{D}, \quad 1 \leqq j \leqq n, \quad 1 \leqq k, r \leqq l_2 , \tag{3.8}$$

where D_{jkr} is the determinant of the matrix obtained by substituting the rth column in $\mathscr{A}$ by the vector $(-L_j^0 g_q^k + \sum_{1 \leqq p \leqq l_1} (L_j^0 f_p^k) R_p g_q^0)$, $1 \leqq q \leqq l_2$. Summing (3.8) over r we obtain

$$\sum_{1 \leqq r \leqq l_2} \mu_{jrr}^0 = D^{-1} \sum_{1 \leqq r \leqq l_2} D_{jrr} = \frac{-L_j^0 D}{D} = -L_j^0 \log D . \tag{3.9}$$

Here we have used the identity $(L_j^0 f_p^k) R_p g_q^0 = L_j^0 (f_p^k R_p g_q^0)$ which is a consequence of $[L_j^0, R_p] = 0$ (cf (2.8)) and $L_j^0 g_q^0 = 0$.

Note that $D(0) \neq 0$. From (3.2) and (3.9) it follows that a desired solution of (3.1) is given by $V = D^{-1}$. This completes the proof of Theorem 1.

§4. Proof of Theorem 2

We shall first show that the distribution solution T of Theorem 1 is not the boundary value of a holomorphic function in any wedge with edge M. We shall then use T to construct a CR function of class C^k with the same property. We use the closed graph theorem, together with a result from [4] to show that there exists a smooth CR function which does not extend.

We review here some basic properties of boundary values of holomorphic functions in wedges. If M is an embedded generic CR manifold of $\mathbf{C}^{n+l}$ of dimension $2n + l$ given by $\operatorname{Im} w = \phi(z, \bar{z}, s)$ as in (2.2), and Γ an open strictly convex cone of $\mathbf{R}^l$ and $\mathcal{O}$ an open neighborhood of 0 in $\mathbf{C}^{n+l}$, we define the wedge $\mathscr{W}(\mathcal{O}, \Gamma)$ by

$$\mathscr{W}(\mathcal{O}, \Gamma) = \{(z, w) \in \mathcal{O}, \operatorname{Im} w - \phi(z, \bar{z}, s) \in \Gamma\} . \tag{4.1}$$

If h is a holomorphic function in $\mathscr{W}(\mathcal{O}, \Gamma)$ with slow growth at the edge M, i.e. $|h(m)| \leqq C \operatorname{dist}(m, M)^{-p}$, for $m \in \mathscr{W}(\mathcal{O}, \Gamma)$, then h has a boundary value $b_M h$ as a CR distribution on $M \cap \mathcal{O}$. Furthermore, if $b_M h = 0$ in an open set of $M \cap \mathcal{O}$ and $\mathscr{W}(\mathcal{O}, \Gamma)$ is connected then h vanishes identically. (For a flat wedge see [7], and for the general case see [1] and [2].)

We begin with the following result.

Proposition 4.2. *Under the assumptions of Theorem* 2, *for any* $k \geqq 0$ *there is a CR function of class* C^k *defined in a neighborhood of* m_0 *which does not extend to any wedge with edge* M *near* m_0.

Proof. The fact that the distribution solution T of Theorem 1 is not the boundary value of a holomorphic function in any wedge with edge M is proved by the following lemma.

Lemma 4.3. *Let M be an embedded generic CR manifold and S a CR distribution on M, nonzero in any neighborhood of m_0, with the following property. For every neighborhood U of m_0 in M there is an open set $U' \subset U$ such that S vanishes in U'. Then in any neighborhood of m_0, S is not the boundary value of a holomorphic function in a wedge with edge M.*

Proof. Indeed, it follows from the remarks preceding Proposition 4.2 that if S were the boundary value of a holomorphic function h in a connected wedge $\mathcal{W}(\mathcal{O}, \Gamma)$, then, since S vanishes in an open subset of any neighborhood of m_0, h would have to vanish identically, which would imply that S is the zero distribution. Since Γ is assumed to be strictly convex, the wedge $\mathcal{W}(\mathcal{O}, \Gamma)$ is connected if $\mathcal{O}$ is connected and sufficiently small. This proves the lemma.

To construct the C^k CR function of the proposition we shall use the following. As in §2 we can find l vector fields D_j satisfying, for $1 \leqq q \leqq n$, $1 \leqq j, p \leqq l$,

$$[L_q, D_j] = 0, \quad [D_j, D_p] = 0, \quad \text{and} \quad D_j(s_p + i\phi_p(z, \bar{z}, s)) = \varepsilon_{jp},$$

where the ϕ_p are as in (2.2).

Lemma 4.4. *Let S be a CR distribution defined in a neighborhood of m_0 in M. Then for any k there is a CR function $f \in C^k$ defined near m_0 such that*

$$S = \left(\sum_{1 \leqq j \leqq l} D_j^2 \right)^p f,$$

where the D_j are the vector fields defined above.

This lemma is a consequence of the representation of distributions annihilated by a system of complex vector fields given in [5] (see also [13]).

We may now prove Proposition 4.2. By Theorem 1 we can find a CR distribution T in an open subset U with support in N. By Lemma 4.4 we may write $T = (\sum_{1 \leqq j \leqq l} D_j^2)^p f$ with f a CR function of class C^k. We reason by contradiction. Assume that f is the boundary value of a holomorphic function H in a wedge $\mathcal{W}(\mathcal{O}, \Gamma)$. Since D_j is of the form

$$D_j = \frac{\partial}{\partial w_j} + \sum_{1 \leqq k \leqq l} a_{jk} \frac{\partial}{\partial \bar{w}_k},$$

(see [3]), we have $b\left(\dfrac{\partial H}{\partial w_j}\right) = D_j(bH)$. Note that $\dfrac{\partial H}{\partial w_j}$ is of slow growth, by Cauchy's inequalities. We conclude

$$D_j f = b\left(\frac{\partial H}{\partial w_j}\right), \quad T = \left(\sum_{1 \leqq j \leqq l} D_j^2 \right)^p f = b\left(\left(\sum_{1 \leqq j \leqq l} \left(\frac{\partial^2}{\partial w_j^2} \right) \right)^p H \right). \tag{4.5}$$

It would then follow from (4.5) that T is the boundary value of a holomorphic function, contradicting the conclusion of Lemma 4.2. The proof of Proposition 4.2 is now complete.

Proof of Theorem 2. We assume by contradiction that every smooth CR function defined near m_0 extends holomorphically in some wedge. We claim that for every

open neighborhood U of $m_0 \in M$ there exists a wedge $\mathscr{W}(\mathscr{O}, \Gamma)$ to which every smooth CR function defined on U extends holomorphically. Indeed, this follows from an argument using the Baire category theorem by a slight modification of the proof of Theorem 7 of [4], where the corresponding result is proved for continuous functions. We shall show that if k is sufficiently large, then any CR function of class C^k defined near m_0 extends to a wedge, contradicting Proposition 4.2.

If U is a neighborhood of m_0, let E be the space of smooth CR functions on U and $\mathscr{W}(\mathscr{O}, \Gamma)$ the wedge associated to U by the claim above and denote by $\mathscr{H}$ the space of bounded holomorphic functions in $\mathscr{W}(\mathscr{O}, \Gamma)$. By the closed graph theorem the mapping $E \to \mathscr{H}$ which to each CR function associates its holomorphic extension is continuous. We can therefore find k, U' relatively compact in U and a constant C such that for all $f \in E$,

$$\sup_{z \in \mathscr{W}(\mathscr{O}, \Gamma)} |\tilde{f}(z)| \leqq C \sup_{\substack{m \in U' \\ |\alpha| \leqq k}} |D^\alpha f(m)| , \tag{4.6}$$

where $\tilde{f}$ is the holomorphic extension of f. Since every CR function of class C^k is locally a limit, in the C^k topology of entire functions [5], it follows from (4.6) that every CR C^k function on U extends holomorphically to $\mathscr{W}(\mathscr{O}, \Gamma)$. Theorem 2 is then proved by contradiction.

§5. Characterization of minimal CR submanifolds; proof of Theorem 3

The proof of the uniqueness in Theorem 3, as well as the characterization in (i), is essentially based on the following localized version of a result of Sussmann [11]: Let $\{X_1, \ldots, X_p\}$ be a set of smooth real vector fields defined near the origin in $\mathbf{R}^q$, and U' a sufficiently small neighborhood of 0. Then there is a unique submanifold $N \subset U'$, $0 \in N$, such that the X_j are all tangent to N at every point of N and for which $\dim N$ is minimal with this property. Its uniqueness follows from the fact that it is the union of all points in U' which can be reached by a finite sequence of integral curves, contained in a slightly bigger neighborhood U, of the vector fields X_j.

We now consider the case of a CR manifold M. We take for $X_1, \ldots, X_{2n}$ a basis for the real and imaginary parts of the local CR vector fields L_j, i.e. for the sections of $\mathrm{Re}\,\mathscr{V}$. A submanifold N of M is a CR submanifold of the same CR dimension if and only if all the X_j are tangent to N. Now the uniqueness and the characterization (i) follow from Sussmann's result stated above.

To show that N_0 can be characterized by (ii), we assume that M is embedded and use the following result of Tumanov [14]: There is a CR submanifold of M through m_0 contained in the union of sets of the form $Z(bD)$, where $Z: \bar{D} \to \mathbf{C}^{n+l}$ is continuous, holomorphic in D and satisfying $m_0 \in Z(bD) \subset U \cap M$, and U is a sufficiently small neighborhood of m_0 in M. We shall show that the image of bD under all such holomorphic discs Z lies in the minimal submanifold N_0. Hence the proof of Theorem 3 will be completed by the following.

Lemma 5.1. *Let M be a generic CR submanifold of $\mathbf{C}^{n+l}$ and N a CR submanifold containing m_0. Then there is a neighborhood U of m_0 in M such that for every*

$Z: D \to C^{n+l}$ with Z holomorphic, continuous in $\bar{D}$ and satisfying $m_0 \in Z(bD) \subset U \cap M$, then $Z(bD) \subset N$.

Proof. We use the coordinates $(x, y, s, t) = (x, y, s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2})$ introduced in §2, so that N is the submanifold $\{t = 0\}$. For a disc Z as in the statement of the lemma, we write $Z(\zeta) = (z(\zeta), w(\zeta))$; we may assume without loss of generality that $Z(1) = m_0$. Then

$$\operatorname{Re} w(\zeta) = -T_1(\phi(z(\cdot), \overline{z(\cdot)}, \operatorname{Re} w(\cdot)))(\zeta), \quad \zeta \in bD\,, \tag{5.2}$$

where ϕ is given by (2.2) and $T_1 f$ is the harmonic conjugate of f, defined on bD vanishing at 1. As in (2.7) we may write

$$\operatorname{Re} w_j(\zeta) = s_j(\zeta), \quad 1 \leqq j \leqq l_1\,,$$
$$\operatorname{Re} w_j(\zeta) = t_j(\zeta) + \psi_j(z(\zeta), \overline{z(\zeta)}, s(\zeta)), \quad 1 \leqq j \leqq l_2\,.$$

Equation (5.2) becomes

$$\begin{aligned} &s_j(\zeta) = -T_1(\phi_j(z, \bar{z}, s, t + \psi(x, y, s)))(\zeta), \quad 1 \leqq j \leqq l_1, \\ &t_j(\zeta) + \psi_j(z(\zeta), \overline{z(\zeta)}, s(\zeta)) = -T_1(\phi_{j+l_1}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)))(\zeta), \quad 1 \leqq j \leqq l_2\,. \end{aligned} \tag{5.3}$$

We claim that (5.3) implies that $t(\zeta) = 0$ for all $\zeta \in bD$, which is the desired conclusion. We shall show first that

$$L_k^0(\psi_j(z, \bar{z}, s) + i\phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s)) = 0, \quad 1 \leqq j \leqq l_2, \quad 1 \leqq k \leqq n\,, \tag{5.4}$$

where L_k^0 is given by (2.6). Indeed, since

$$L_k g_j = 0, \quad 1 \leqq j \leqq l_2, \quad 1 \leqq k \leqq n\,, \tag{5.5}$$

where g_j is given in (2.7), we obtain (5.4) by putting $t = 0$ in (5.5). Now the claim follows from a standard approximation argument as follows. Since $g_j^0 = \psi_j(z, \bar{z}, s) + i\phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s))$ is a CR function for the induced CR structure on N, it is the uniform limit of holomorphic polynomials [5]. By the maximum principle, the pullback of g_j^0 to bD by the map $(z(\zeta), s(\zeta))$ extends holomorphically to D. Therefore we have, since $\psi_j(0) = 0$,

$$\psi_j(z(\zeta), \overline{z(\zeta)}, s(\zeta)) = -T_1(\phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s)))(\zeta)\,,$$

which proves that $t(\zeta) \equiv 0$ is a solution of the second equation in (5.3). Since the map

$$t \to T_1[\phi_{j+l_1}(z, \bar{z}, s, t + \psi(z, \bar{z}, s)) - \phi_{j+l_1}(z, \bar{z}, s, \psi(z, \bar{z}, s))]$$

is a contraction in $L^2(bD)$ if the neighborhood U of the lemma is sufficiently small, the solution is unique, which proves the claim, and completes the proof of the lemma.

§6. Other results and remarks

Inspection of the proof of Theorem 1 shows that its conclusion holds under assumptions weaker than embeddability of M. We can also weaken the smoothness

assumption for M. If N is a CR submanifold of an abstract CR manifold of class C^2 we shall say that M is *locally embeddable of order* 1 *along* N if around every point $m_0 \in N$ there exist $n + l$ functions Z_j of class C^2 such that for every L section of $\mathcal{V}$

$$LZ_j(m) = O(\text{dist}(m, N)^2) .$$

Theorem 4. *The conclusion of Theorem* 1 *holds if* M *is of class* C^2 *and locally embeddable of order* 1 *along* N *or if* $\dim N = 2n$. *In addition, without assuming embeddability, if* N *is of codimension* 1 *in* M *then the function which is identically equal to* 1 *on one side of* N *in* M *and equal to* 0 *on the other side is a singular CR function.*

Proof. As mentioned above, the conclusion when M is locally embeddable of order 1 along N follows by inspection of the proof of Theorem 1. To prove the statement when $\dim N = 2n$, we choose coordinates $u = (u_1, \ldots, u_{2n})$, $t = (t_1, \ldots, t_l)$ such that N is given by $t = 0$. We may write the L_j in the form

$$L_j = \sum_{k=1}^{2n} \alpha_{jk}(u, t) \frac{\partial}{\partial u_k} + \sum_{1 \leqq k, r \leqq l} \mu_{jkr}(u, t) t_k \frac{\partial}{\partial t_r} . \tag{6.1}$$

By the Newlander-Nirenberg Theorem [10], we may find coordinates $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ such that after a linear transformation on the basis L_j we have

$$L_j^0 = \sum_{k=1}^{2n} \alpha_{jk}(u, 0) \frac{\partial}{\partial u_k} = \frac{\partial}{\partial \bar{z}_j} .$$

We then have

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{\substack{1 \leqq k \leqq n \\ 1 \leqq r \leqq l}} \lambda_{jkr} t_r \frac{\partial}{\partial \bar{z}_k} + \sum_{\substack{1 \leqq k \leqq n \\ 1 \leqq r \leqq l}} \nu_{jkr} t_r \frac{\partial}{\partial z_k} + \sum_{1 \leqq k, r \leqq l} \mu_{jkr} t_k \frac{\partial}{\partial t_r} . \tag{6.2}$$

As in the proof of Theorem 1, we look for a solution of the form $V(x, y)\delta(t)$. For this, it suffices to solve the system

$$L_j^0 V + \sum_{1 \leqq r \leqq l_2} \mu_{jrr}^0(z, \bar{z}, s) V = 0, \quad 1 \leqq j \leqq n . \tag{6.3}$$

The equations (6.3) can be solved since the compatibility conditions needed for the existence of a solution follow from the commutation relations $[L_j, L_k] = 0$ for all j, k.

Finally, we consider the case where $\text{codim}_M N = 1$. Since the L_j are tangent to N, the Heaviside function H which is equal to 1 on one side of N and 0 on the other side satisfies $L_j H = 0$ in the distribution sense. This completes the proof of Theorem 3.

Remark 6.4. The conclusion of Theorem 1 need not hold without the embeddability condition, as shown by the following example. Let $M = \mathbf{R}^4$ with coordinates (x, y, s, t) and let

$$L = \frac{\partial}{\partial \bar{z}} - iz \frac{\partial}{\partial s} + \mu(z, \bar{z}, s) t \frac{\partial}{\partial t} , \tag{6.5}$$

with μ a smooth function in $\mathbf{R}^3$ not in the range of the Lewy operator $L^0 = \dfrac{\partial}{\partial \bar{z}} - iz\dfrac{\partial}{\partial s}$ (see [8]). It is clear that finding a solution of the form $V(x, y, s)\delta(t)$, with $V(0) \neq 0$ is equivalent to solving the equation $L^0 V - \mu V = 0$. Note that in this example the Heaviside function $H(t)$ is a singular solution.

Proposition 6.6. *If M is a real analytic generic manifold which is not minimal at m_0, and N a real analytic CR submanifold of M, then there is a holomorphic submanifold $\mathscr{H}$ in $\mathbf{C}^{n+l}$, such that $N = M \cap \mathscr{H}$ with $\dim_{\mathbf{R}} N = \dim_{\mathbf{C}} \mathscr{H} + n$. In particular, the minimal CR submanifold through m_0 is of this form.*

Proof. We choose coordinates (x, y, s, t) as in §2. Here the ϕ_j and ψ_k as in (2.7) are real analytic functions in a neighborhood of 0. Since $L_j^0 g_k^0 = 0$ and the g_k^0 are real analytic, there exists a holomorphic function $G_k(z, w_1, \ldots, w_{l_1})$ such that g_k^0 is the restriction of G_k to N. In the new variables $w_j' = w_j$, $1 \leqq j \leqq l_1$, and $w_{j+l_1}' = w_{j+l_1} - G_j(z, w_1, \ldots, w_{l_1})$, $1 \leqq j \leqq l_2$, we have $N = M \cap \{w_{j+l_1}' = 0,\ 1 \leqq j \leqq l_2\}$. This proves the first statement. To prove the second claim, it suffices to observe that when M is real analytic, the minimal CR submanifold through m_0 is the Nagano leaf (see [9]) of the sections of $\mathrm{Re}\,\mathscr{V}$, which is real analytic.

Remark 6.7. If M is assumed only to be smooth rather than real analytic, the minimal CR submanifold N need not be of the form $N = M \cap \mathscr{H}$ with $\mathscr{H}$ a holomorphic submanifold. Indeed, consider the generic submanifold of $\mathbf{C}^3$ parametrized by (x, y, s, t) and given by $\{(z, w_1, w_2)\colon w_1 = s + i|z|^2, w_2 = t + h(x, y, s)\}$ where h is a smooth non real analytic function satisfying $\dfrac{\partial h}{\partial \bar{z}} = iz\dfrac{\partial h}{\partial s}$. Here N is given by $\{t = 0\}$ and $L = \dfrac{\partial}{\partial \bar{z}} - iz\dfrac{\partial}{\partial s}$. If N were the intersection of M with a complex hypersurface, there would exist a holomorphic function $\mathscr{K}$ and holomorphic coordinates (z', w_1', w_2') such that h is the restriction of $\mathscr{K}(z', w_1', w_2')$ to N, contradicting the assumption that h is not real analytic.

References

1. Baouendi, M.S., Chang, C.H., Treves, F.: Microlocal hypo-analyticity and extension of CR functions. J. Differ. Geom. **18**, 331–391 (1983)
2. Baouendi, M.S., Rothschild, L.P.: Normal forms for generic manifolds and holomorphic extension of CR functions. J. Differ. Geom. **25**, 431–467 (1987)
3. Baouendi, M.S., Rothschild, L.P.: Embeddability of abstract CR structures and integrability of related systems, Ann. Inst. Fourier **37**, 131–141 (1987)
4. Baouendi, M.S., Rothschild, L.P.: Extension of holomorphic functions in generic wedges and their wave front sets. Commun. Partial Differ. Equations **13**, 1441–1466 (1988)
5. Baouendi, M.S., Treves, F.: A property of the functions and distributions annihilated by a locally integrable system of complex vector fields. Ann. Math. **113**, 387–421 (1981)
6. Bloom, T., Graham, I.: On 'type' conditions for generic submanifolds of C^n. Invent. Math. **40**, 217–243 (1977)
7. Hörmander, L.: The Analysis of Linear Partial Differential Operators, I. Berlin-Heidelberg-New York: Springer 1983

8. Lewy, H.: On the local character of the solution of an atypical differential equation in three variables and a related problem for regular functions of two complex variables. Ann. Math. **64**, 514–522 (1956)
9. Nagano, T.: Linear differential systems with singularities and an application to transitive Lie algebras. J. Math. Soc. Japan **18**, 398–404 (1966)
10. Newlander, A., Nirenberg, L.: Complex coordinates in almost complex manifolds. Ann. Math. **65**, 391–404 (1957)
11. Sussmann, H.J.: Orbits of families of vector fields and integrability of distributions. Trans. Am. Math. Soc. **180**, 171–188 (1973)
12. Trepreau, J.M.: Sur le prolongement holomorphe des fonctions CR définis sur une hypersurface réelle de classe C^2 dans $\mathbf{C}^n$. Invent. Math. **83**, 583–592 (1986)
13. Treves, F.: Approximation and representation of functions and distributions annihilated by a system of complex vector fields. Ecole Polytechnique, France (1981)
14. Tumanov, A.E.: Extending CR functions on manifolds of finite type to a wedge (Russian). Math. Sb. Nov. Ser. **136**, 128–139 (1988)
15. Zachmanoglou, E.C.: Solutions of partial differential equations with support on leaves of associated foliations. Trans. Am. Math. Soc. **180**, 415–421 (1973)

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A Gauss-Bonnet theorem for motivic cohomology

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Introduction

Let F be a totally real number field, G be a semi-simple F-algebraic group, $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$ be the group of real points, K be a maximal compact subgroup of G_∞ and $X = K \backslash G_\infty$ be the associated symmetric space. Choose a G_∞-invariant Riemannian metric on X and let ω_X be its curvature form. There exists a unique Haar measure μ on G_∞ such that the image of μ on the homogeneous space X is the measure defined by ω_X, [8] §3.2. Let Γ be a torsion free arithmetic subgroup of $G(F)$, then X/Γ is a manifold with finite volume and it follows easily from the Gauss-Bonnet-Allendoerfer-Weil-Chern theorem that $\int_{G_\infty/\Gamma} \mu = \chi(X/\Gamma)$, in case X/Γ is compact, [8] Prop. 22. Remarkably, this formula for the Euler characteristic of X/Γ continues to hold even without the compactness hypothesis, but this is a much deeper result, [3], [8] §3.6.

The purpose of this paper is to show that there is an analogous formula for the arithmetic Euler-Poincaré characteristic of a smooth, projective surface $\mathscr{E}$ defined over a finite field and possessing a fibration by curves. This Euler characteristic arises from the étale cohomology of the constant sheaf $\mathbb{Z}$ and is thus intrinsic to the surface. The Gauss-Bonnet formula shows that it can basically be expressed in measure theoretic terms. (See Section 2 for the statement and proof of the theorem.) On the other hand, just as the form ω_X and the measure μ depend on a choice of a Riemannian structure on X, the measures and the arithmetically defined lattices associated to $\mathscr{E}$ are constructed using a specific fibration. Their definition and the calculation of the covolumes associated to the fibers are given in Section 1. Combining this computation with an L-series argument gives the asymptotic behavior of the product of these covolumes.

The definition of the Euler characteristic $\chi(\mathscr{E}, \mathbb{Z})$ and Milne's theorem relating its value to the residue of the zeta function of the surface at $s = 0$ are recalled at the beginning of Section 2. The Gauss-Bonnet formula then follows easily from his

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result and the calculations in the previous section. The formula can be given quite explicitely when $\mathscr{E}$ is the complete Néron model of an elliptic curve defined over a function field. In Section 3 this is worked out in detail for the case of a semi-stable fibration.

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§1. Let C be a complete, smooth, geometrically irreducible curve over $\mathbb{F}_q$ and L be the function field of C. Let E be a curve over L that is also complete, smooth, and geometrically irreducible. Then there exists a regular curve $\mathscr{E}$ proper, flat and of finite type over C such that the generic fiber is L-isomorphic to E ([1] Resolution Theorem). $\mathscr{E}$ is smooth projective surface over $\mathbb{F}_q$. Let $p\colon \mathscr{E} \to C$ be the $\mathbb{F}_q$-rational projection. We assume there is an $\mathbb{F}_q$-rational section $\sigma\colon C \to \mathscr{E}$.

For each place v of L let $q_v = q^{\deg v}$. For all but a finite number of v, $p^{-1}(v)$ is smooth and may be regarded as a curve over $\mathbb{F}_{q_v}$. The other fibers may be singular or even reducible. In this section we define the measures and lattices associated to the fibers, give a formula for the covolumes of the lattices, and evaluate the limit of the product of these constants. For the convenience of the reader, basic facts concerning the Tamagawa measure of SL_2 and standard volume computations are recalled in an appendix. Everything mentioned there is contained in [10] or in [11].

Let $O_{\mathscr{E}, p^{-1}(v), \sigma}$ denote the local ring of the component of the fiber over v containing σv and $k(\mathscr{E}_{v,\sigma})$ be its residue field; $k(\mathscr{E}_{v,\sigma})$ is isomorphic to the function field of this component of $p^{-1}(v)$ and has field of constants $\mathbb{F}_{q_v}$.

Definition. For each place v of L, let τ_v denote the global Tamagawa measure for $SL_2(k(\mathscr{E}_{v,\sigma}))$ and set $\mu_v = q_v^{3g_v-3}(1 - q_v^{-1})\tau_v$, where g_v is the genus of $k(\mathscr{E}_{v,\sigma})$.

Remark. The factor $q_v^{3g_v-3}(1 - q_v^{-1})$ plays a role analogous to the convergence factors introduced by Weil in the study of affine algebraic groups, [11].

Let $k(\mathscr{E}_{v,\sigma})_{\sigma v}$ be the completion at σv. This locally compact field with residue field $\mathbb{F}_{q_v}$ is isomorphic to $\mathbb{F}_{q_v}((T))$ and thus also isomorphic to L_v. For each v choose an embedding $i_v\colon k(\mathscr{E}_{v,\sigma}) \to L_v$, continuous for the σv-adic topology, of the global field into L_v. Though there is no canonical way to do this, i_v provides a metric on the functions of $k(\mathscr{E}_{v,\sigma})$. Another choice of the embedding would give the same metric. This procedure is analogous to the choice of a Riemannian structure on the symmetric space X mentioned in the introduction and gives volumes associated to each fiber of p.

Definition. Let $\mathfrak{a}_{(\sigma v)}$ be the subring of $k(\mathscr{E}_{v,\sigma})$ consisting of functions which are integral at all places of $k(\mathscr{E}_{v,\sigma})$ except σv and set $\Gamma_v = SL_2(i_v(\mathfrak{a}_{(\sigma v)}))$.

Proposition. *For a place v of L, let $M_v = \mu_v(SL_2(L_v)/\Gamma_v)$, then $M_v = (1 - q_v^{-2})(1 - q_v^{-1})\zeta_{k(\mathscr{E}_{v,\sigma})}(2)$.*

Proof. The proposition follows immediately from the definitions and the lemma in the appendix.

Definition. For a finite set of points $S \subseteq C$ let $U = C - S$ and $L_{\mathbb{A}}(S)$ be the ring of S-adeles. Let $\mathscr{G}_S = SL_2(L_{\mathbb{A}}(S)) = \prod_{v \in S} SL_2(L_v) \times \prod_{v \in U} SL_2(r_v)$, where r_v denotes the ring of integers in L_v, and μ_S be the measure $\dfrac{hq^{1-g}}{q-1} \prod_{v \in S} \mu_v \times \prod_{v \in U} \omega_v$, with $h = [\mathrm{Jac}(C)(\mathbb{F}_q)]$ and with $g =$ genus C. Let $\mathscr{H}_S$ be the lattice $\prod_{v \in S} \Gamma_v \times \prod_{v \in U} I_v$ in $\mathscr{G}_S$, where $I_v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_S = \mu_S(\mathscr{G}_S / \mathscr{H}_S)$.

Lemma. *Let M_v denote the covolume given in the proposition, then ΠM_v converges to $\dfrac{(q-1)\log q}{h\, q^{1-g}} \zeta(C,2)^{-1} \cdot \Pi K_v \cdot \mathrm{Res}_{s=2} \zeta(\mathscr{E}, s)$, where $K_v = \zeta_{k(\mathscr{E}_v, o)}(2) \zeta(\mathscr{E}_v, 2)^{-1}$ and the product extends over all v such that $p^{-1}(v)$ is singular.*

Proof. It follows from the Riemann hypothesis that the function $L(s)^{-1} = \zeta(C,s)^{-1}$ $\zeta(C, s-1)^{-1}$ $\zeta(\mathscr{E}, s)$ is holomorphic for $\mathrm{Re}\ s > 3/2$. It has value $\dfrac{(q-1)\log q}{h\, q^{1-g}}$ $\zeta(C,2)^{-1}$ $\mathrm{Res}_{s=2}\, \zeta(\mathscr{E}, s)$ at $s = 2$. Since $\zeta(\mathscr{E}, s)$ is given by the absolutely convergent product $\prod_{v \in C} \zeta(\mathscr{E}_v, s)$ for $\mathrm{Re}\ s > 2$, $L(s)^{-1} = \prod_{v \in C} (1 - q_v^{-s})(1 - q_v^{1-s})\, \zeta(\mathscr{E}_v, s)$ in that half-plane, but in fact this product converges absolutely for $\mathrm{Re}\ s > 3/2$ and thus takes the value given above at $s = 2$. By the proposition the factor in $L(2)^{-1}$ equals M_v for all $v \in C$ such that $p^{-1}(v)$ is smooth; the lemma now follows from that proposition.

Proposition. *Let $M = \lim_S M_S$, where the limit is taken over increasing finite subsets S of C, then $M =$ $(\log q)$ $\zeta(C,2)^{-1} \cdot \Pi K_v \cdot \mathrm{Res}_{s=2}\, \zeta(\mathscr{E}, s)$.*

Proof. By definition, $M_S = \dfrac{hq^{1-q}}{q-1} \prod_{v \in S} M_v \times \prod_{v \in U} (1 - q_v^{-2})$. For a sufficiently large finite set S, M_s approximates $\prod_v M_v$ arbitrarily closely so the proposition follows from the lemma.

Definition. The constant M will be called the *covolume* of the system of triples $(\mathscr{G}_S, \mathscr{H}_S, \mu_S)$.

Appendix

Let K be a global field of genus g with field of constants $\mathbb{F}_\ell$, $G = SL_2(K)$, $G_{\mathbb{A}}$ be the adelization of G and G_K be the principal elements. For a place v of K denote by R_v and $\mathbb{F}_{\ell_v}$, respectively, the ring of integers and residue field of K_v. Recall that the local Tamagawa measure ω_v on $SL_2(K_v)$ is the Haar measure which gives the compact subgroup $SL_2(R_v)$ volume $1 - \ell_v^{-2}$ and that the global Tamagawa measure τ on $G_{\mathbb{A}}$ is ℓ^{3g-3} times the product measure $\Pi\omega_v$. G_K is a discrete subgroup of $G_{\mathbb{A}}$ and $\tau(G_{\mathbb{A}}/G_K)$, the Tamagawa number of G, is one, [10], [11].

Let $\mathfrak{a}_{(v)}$ denote the subring of K consisting of elements which are integral at all places w of K, $w \neq v$, $O = M_2(\mathfrak{a}_{(v)})$, $O^1 = SL_2(\mathfrak{a}_{(v)})$, $O_w = O \otimes_{\mathfrak{a}_{(v)}} R_w$, $O_w^1 = SL_2(R_w)$ and $C = \prod_{w \neq v} O_w^1$. Let $U = SL_2(K_v) \times C$. U is an open subgroup of $G_{\mathbb{A}}$,

$G_{\mathbb{A}} = G_K U$, $G_K \cap U = O^1$, hence U/O^1 is isomorphic to $G_{\mathbb{A}}/G_K$ and O^1 is a discrete subgroup of U with covolume one.

Let p denote the projection of O^1 on the factor $SL_2(K_v)$ of U. By [10] IV §1, Lemme 1.2, $p(O^1)$ is a discrete subgroup of $SL_2(K_v)$ with finite covolume. Clearly U/O^1C and $SL_2(K_v)/p(O^1)$ are isomorphic. Let τ denote the Tamagawa measure transported to $SL_2(K_v)/p(O^1)$.

Lemma. $\tau(SL_2(K_v)/p(O^1)) = \ell^{3-3g}(1-\ell_v^{-2})\zeta_K(2)$.

Proof. By the isomorphisms above and the fact that $\tau(U/O^1) = 1$, $\tau(SL_2(K_v)/p(O^1)) = \tau(C)^{-1} = \tau(\prod_{w \neq v} O_w^1)^{-1}$. As $O_w^1 = SL_2(R_w)$, $\tau(\prod_{w \neq v} O_w^1)^{-1} = \ell^{3-3g}\prod_{w \neq v}(1-\ell_w^{-2})^{-1}$.

§2. Let $H^i(\mathscr{E}, \mathbb{Z})$ denote the étale cohomology group of the constant sheaf $\mathbb{Z}$. Then $H^0(\mathscr{E}, \mathbb{Z}) \cong \mathbb{Z}$ and $H^i(\mathscr{E}, \mathbb{Z})$ is torsion for $i > 0$, [5], Cor. 7.2. For $i > 3$, $H^i(\mathscr{E}, \mathbb{Z})$ is finite and is isomorphic to $H^i(\mathscr{E}, \hat{\mathbb{Z}}) = \prod_l H^i(\mathscr{E}, \mathbb{Z}_\ell)$, where the product is taken over all primes ℓ including $\ell = p$, [5] Cor. 7.6(a). Hence $H^i(\mathscr{E}, \mathbb{Z}) = 0$ for $i > 5$, [4] VI, Cor. 1.4 and Remark 1.5(b). $H^1(\mathscr{E}, \mathbb{Z})$ is also finite, [5] Lemma 9.1, and $H^2(\mathscr{E}, \mathbb{Z}) \cong H^2(\mathscr{E}, \mathbb{Z})_{cotor} \oplus \mathbb{Q}/\mathbb{Z}$ where $H^2(\mathscr{E}, \mathbb{Z})$ is the cotorsion subgroup of $H^2(\mathscr{E}, \mathbb{Z})$, [5] Remark 7.7(a). Let

$$\chi(\mathscr{E}, \mathbb{Z}) = \frac{[H^0(\mathscr{E}, \mathbb{Z})_{tor}][H^2(\mathscr{E}, \mathbb{Z})_{cotor}][H^4(\mathscr{E}, \mathbb{Z})]}{[H^1(\mathscr{E}, \mathbb{Z})][H^3(\mathscr{E}, \mathbb{Z})][H^5(\mathscr{E}, \mathbb{Z})]},$$

where $[G]$ denotes the order of a group G.

Theorem 0.4 (a) of [5] asserts that $\lim_{s \to 0}(1-q^{-s})\zeta(\mathscr{E}, s) = \pm\chi(\mathscr{E}, \mathbb{Z})$. (Note the obvious misprint in the statement on p. 301). The functional equation for $\zeta(\mathscr{E}, s)$ takes the form $\zeta(\mathscr{E}, 2-s) = \pm q^{(1-s)\chi}\zeta(\mathscr{E}, s)$, where χ is the self-intersection number of the diagonal on $\mathscr{E} \times \mathscr{E}$. (The number χ may also be interpreted as the second Chern class of the tangent bundle on $\mathscr{E}$ or as the Euler-Poincaré characteristic in ℓ-adic étale, $\ell \neq p$, or in crystalline cohomology, [4] V, Thm. 3.12, [5] § 10.)

Theorem. *Let M be the covolume of the system $(\mathscr{G}_s, \mathscr{H}_s, \mu_s)$, and for $v \in C$ such that $p^{-1}(v)$ is singular let K_v be the constant defined in the lemma of Section 1, then* $\chi(\mathscr{E}, \mathbb{Z}) = M\, q^{-\chi}\, \zeta(C, 2)\, \Pi\, K_v^{-1}$.

Proof. Since both sides of the formula are positive, the ambiguities of the signs in Milne's theorem and in the functional equation will be disregarded in the proof. Let $L(s)$ be the function defined in the proof of the lemma in Section 1. In the half-plane Re $s < 1/2$, $L(2-s)^{-1}$ is given by the absolutely convergent product $\prod_{v \in C}(1-q_v^{s-2})\,(1-q_v^{s-1})\,\zeta(\mathscr{E}_v, 2-s)$. By the proof of that lemma and the proposition following it, this product converges to $\dfrac{q-1}{hq^{1-g}}\, M \cdot \Pi K_v^{-1}$ as s approaches zero. On the other hand, $L(2-s)^{-1} = \zeta(C, 2-s)^{-1}\,\zeta(C, 1-s)^{-1}\,q^{\chi(1-s)}\,\zeta(\mathscr{E}, s)$. Write $\zeta(C, s) = P_C(q^{-s})(1-q^{-s})^{-1}(1-q^{1-s})^{-1}$, then $L(2-s)^{-1} = \zeta(C, 2-s)^{-1}\, P_C(q^{s-1})^{-1}\,(1-q^{s-1})(1-q^s)\,q^{\chi(1-s)}\,\zeta(\mathscr{E}, s)$, so $L(2)^{-1} = \zeta(C,\ 2)^{-1}\ q^g h^{-1}(1-q^{-1})q^{\chi}\,\chi(\mathscr{E}, \mathbb{Z})$. The theorem now follows by comparing the two expressions for $L(2)$.

It is rather surprising that three of the terms appearing in the theorem have interpretations in K-theory. The constant sheaf $\mathbb{Z}$ is the sheaf on $\mathscr{E}_{\acute{e}t}$ associated with the presheaf $U \to K_0(\Gamma(U, \mathcal{O}_\mathscr{E}))$ and it is from this point of view that Milne [5] §7 studies its cohomology; in other words, $\chi(\mathscr{E}, \mathbb{Z})$ really means $\chi(\mathscr{E}, K_0\, \mathcal{O}_\mathscr{E})$. The factor $\zeta(C, 2)$ has value $q^{-3g}\,(1-q^{-2})^{-1}\,(1-q^{-1})^{-1}$ [Ker λ_L], where λ_L: $K_2(L) \to \coprod_v \mathbb{F}_{q_v}^*$ is the tame symbol and g is the genus of C [9] §4. Finally, let $K_{m,p}$ be the two dimensional local fields associated to $\mathscr{E}$ by Parshin [7] §2. There is an adele group $I_\mathscr{E} = \prod'_{m,p} K_2^{top}(K_{m,p})$ and a homomorphism $|\ |: I_\mathscr{E} \to \mathbb{R}$. He then associates a class of adeles to differential forms on $\mathscr{E}$ and proves that for any adele $\mathbf{a}$ in this class $|\mathbf{a}| = q^\chi$.

§3. In this section let $\mathscr{E}$ be the complete Néron model of an elliptic curve over L and assume that the fibers of the projection p are semi-stable. In this case the term ΠK_v appearing in the theorem in Section 2 can be easily computed. Recall that the component of a singular fiber containing σv is either a rational curve with a node ([6] III §10, case b_1) or a projective line (*loc. cit.* case b_m, $m > 1$), so in both cases the function field $k(\mathscr{E}_{v,\sigma})$ of Section 1 has genus zero; since there are no non-trivial Severi-Brauer varieties over a finite field, it is a field of rational functions and $\zeta_{k(\mathscr{E}_{v,\sigma})}(s) = (1 - q_v^{-s})^{-1}\,(1 - q_v^{1-s})^{-1}$.

The zeta function of $\mathscr{E}_v$ itself can be found by examining the fiber and its cohomology. An $\mathbb{F}_{q_v}$-irreducible component $\mathscr{E}_{v,a}$ of $\mathscr{E}_v$ may split into distinct irreducible components over $\bar{\mathbb{F}}_{q_v}$. Let $c_{v,a}$ be the number of these components and $P_\ell(\mathscr{E}_v, T)$ denote the characteristic polynomial of the geometric frobenius morphism acting on the ℓ-adic étale cohomology group $H^1(\bar{\mathscr{E}}_v, \mathbb{Q}_\ell)$. The zeta function of $\mathscr{E}_v$ is

$$Z(\mathscr{E}_v, T) = \frac{P_\ell(\mathscr{E}_v, T)}{(1-T)\,\Pi\,(1-(q_v T)^{c_{v,a}})},$$

where the product extends over all $\mathbb{F}_{q_v}$-irreducible components of $\mathscr{E}_v$ and $T = q_v^{-s}$, [2] Prop. 3.3. (Note the misprint in the statement, though not in the proof, on p. 176.)

The set of smooth points of the component of $\mathscr{E}_v$ containing σv form an $\mathbb{F}_{q_v}$-torus that may or may not be split. In case b_1, $\mathscr{E}_v$ is geometrically irreducible and $P_\ell(\mathscr{E}_v, T)$ equals $1 - T$, if the torus is split and $1 + T$, if it's twisted. In case b_m, $m > 1$, $\mathscr{E}_v$ consists of m_v projective lines, each with self-intersection -2 and forming a "polygon". In the split situation, each of these components is $\mathbb{F}_{q_v}$-rational, so the product in the denominator of $Z(\mathscr{E}_v, T)$ has value $(1 - q_v T)^{m_v}$. It remains to treat the case b_m, $m > 1$, in case the torus is non-split. Here it makes a difference whether m_v is even or odd. If m_v is even, two of the projective lines are $\mathbb{F}_{q_v}$-rational (the one containing σv and the one of order two in the cyclic group formed by the components) and the other $m_v - 2$ occur in $\mathbb{F}_{q_v^2}$-conjugate pairs. Thus, the product in the denominator of $Z(\mathscr{E}_v, T)$ equals $(1 - q_v T)^2\,(1 - (q_v T)^2)^{(m_v-2)/2}$. For m_v odd, the line containing σv is $\mathbb{F}_{q_v}$-rational, the other $m_v - 1$ appear as $\mathbb{F}_{q_v^2}$-conjugate pairs and the product has value $(1 - q_v T)(1 - (q_v T)^2)^{(m_v-1)/2}$. Since $\Pi K_v =$

$\Pi(1 - q_v^{-2})^{-1}\ (1 - q_v^{-1})^{-1}\ \zeta(\mathscr{E}_v, 2)^{-1}$, it is determined by these expressions for $Z(\mathscr{E}_v, T)$.

The value of ΠK_v can also be calculated for the Néron model of an elliptic curve when there are fibers of additive type. This is done by carrying out the same kind of case by case analysis as in the semi-stable situation, but based instead on the tables in [6] III §11–15.

References

1. Abhyankar, S.: Resolution of singularities of arithmetical surfaces. (Arithmetical Algebraic Geometry, Schilling, O.F.G. (ed.) pp. 111–152). New York: Harper and Row 1965
2. Gordon, W.J.: Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer. Compositio Math. **38**, 163–199 (1979)
3. Harder, G.: A Gauss-Bonnet formula for discrete arithmetically defined groups. Ann. Sc. Éc. Norm. Super. IV. Ser. **4**, 409–455 (1971)
4. Milne, J.S.: Étale cohomology. Princeton: Princeton University Press 1980
5. Milne, J.S.: Values of zeta functions of varieties over finite fields. Am. J. Math. **108**, 297–360 (1986)
6. Néron, A.: Modèles minimaux des varietés abéliennes sur les corps locaux et globaux. Publ. Math., Inst. Hautes Étud. Sci. **21**, 361–484 (1964)
7. Parshin, A.N.: Chern classes, adeles and L-functions. J. Reine Angew. Math. **341**, 174–192 (1983)
8. Serre, J.P.: Cohomologie des groupes discretes. (Oeuvres Collected Papers, Vol. II, pp. 593–685). Berlin-Heidelberg-New York-Tokyo: Springer 1986
9. Tate, J.: Symbols in arithmetic. (Actes, Congrès. Intern. Math., 1970, Tome I, pp. 201–211). Paris: Gauthier-Villars 1971
10. Vignéras, M.F.: Arithmétique des Algèbres de Quaternions. (Lecture Notes in Mathematics, vol. 800). Berlin-Heidelberg-New York: Springer 1980
11. Weil, A.: Adeles and Algebraic Groups. Notes by M. Demazure and T. Ono. (Progress in Math., vol. 23). Boston: Birkhäuser 1982

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Exponential sums on $(\mathbf{G}_m)^n$

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§1. Introduction

Let p be a prime number, $\mathbf{F}_q$ the finite field of $q = p^a$ elements. Let f be an $\mathbf{F}_q$-regular function on $(\mathbf{G}_m)^n$ and write

$$f = \sum_{j \in J} a_j x^j \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}], \tag{1.1}$$

where the sum is over a finite subset J of $\mathbf{Z}^n$. We assume $a_j \neq 0$ for all $j \in J$. Let $\Psi: \mathbf{F}_q \to \mathbf{Q}(\zeta_p)$ be a nontrivial additive character (where ζ_p is a primitive p-th root of unity) and set

$$S_i(f) = \sum_{x_1, \ldots, x_n \in \mathbf{F}_{q^i}^\times} \Psi \circ \mathrm{Tr}_i(f(x_1, \ldots, x_n)) \in \mathbf{Q}(\zeta_p), \tag{1.2}$$

where $\mathrm{Tr}_i: \mathbf{F}_{q^i} \to \mathbf{F}_q$ is the trace map. The corresponding L-function is

$$L(f, t) = \exp\left(\sum_{i=1}^{\infty} S_i(f) t^i / i\right) \in \mathbf{Q}(\zeta_p)[[t]]. \tag{1.3}$$

By well-known theorems of Dwork and Grothendieck, $L(f, t)$ is a rational function, say,

$$L(f, t) = \frac{\Pi_j (1 - \alpha_j t)}{\Pi_l (1 - \beta_l t)}. \tag{1.4}$$

Thus

$$S_i(f) = \sum_l \beta_l^i - \sum_j \alpha_j^i, \tag{1.5}$$

so the problem of estimating $|S_i(f)|$ can be reduced to the problem of estimating the $|\alpha_j|$ and $|\beta_l|$.

Define the *Newton polyhedron* of f, denoted $\Delta(f)$, to be the convex hull in $\mathbf{R}^n$ of the set $J \cup \{(0, \ldots, 0)\}$. Given a face σ of $\Delta(f)$ that contains the origin, let $L(\sigma)$ be

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the smallest linear subspace of $\mathbf{R}^n$ containing σ. We define the *dimension* $\dim \sigma$ of σ to be the dimension of $L(\sigma)$, and we define $\tilde{V}(\sigma)$ to be the volume of σ relative to the induced lattice, i.e., $\tilde{V}(\sigma)$ is the volume of σ with respect to Haar measure on $L(\sigma)$ normalized so that a fundamental domain for the lattice $L(\sigma) \cap \mathbf{Z}^n$ has volume 1. This is easily computable as follows. By simplicial subdivision, one may reduce to the case where σ is a k-simplex with one vertex at the origin and k vertices in $\mathbf{Z}^n$. Let $M(\sigma)$ be the $(k \times n)$-matrix whose rows are the coordinates of the nontrivial vertices of σ. Then $k!\,\tilde{V}(\sigma)$ is the greatest common divisor of the $(k \times k)$-minors of $M(\sigma)$. When the origin is itself a vertex, so that $\sigma_0 = (0, \ldots, 0)$ is a face of $\Delta(f)$, we set $\tilde{V}(\sigma_0) = 1$.

In particular, we regard the entire Newton polyhedron $\Delta(f)$ as a face of itself. We denote $\tilde{V}(\Delta(f))$ simply by $\tilde{V}(f)$. Let $V(f)$ be the volume of $\Delta(f)$ with respect to Lebesgue measure on $\mathbf{R}^n$. Thus

$$V(f) = \begin{cases} \tilde{V}(f) & \text{if } \dim \Delta(f) = n \\ 0 & \text{if } \dim \Delta(f) < n. \end{cases}$$

For any face σ of $\Delta(f)$, let

$$f_\sigma = \sum_{j \in \sigma \cap J} a_j x^j . \tag{1.6}$$

Recall that f is *nondegenerate* (*with respect to its Newton polyhedron*) if for every face σ of $\Delta(f)$ that does not contain the origin, $\partial f_\sigma/\partial x_1, \ldots, \partial f_\sigma/\partial x_n$ have no common zero in $(\bar{\mathbf{F}}_q^\times)^n$, where $\bar{\mathbf{F}}_q$ is the algebraic closure of $\mathbf{F}_q$.

We proved in [1] that if f is nondegenerate and $\dim \Delta(f) = n$, then $L(f, t)^{(-1)^{n-1}}$ is a polynomial of degree $n!\,V(f)$:

$$L(f, t)^{(-1)^{n-1}} = \prod_{i=1}^{n!\,V(f)} (1 - \rho_i t) . \tag{1.7}$$

When p is sufficiently large relative to $\Delta(f)$, we showed in addition that

$$L(f, t)^{(-1)^{n-1}} = \det(I - tF | H_c^n((\mathbf{G}_m)^n \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q, \mathscr{L}_\psi(f))) , \tag{1.8}$$

where $\mathscr{L}_\psi(f)$ is the lisse, rank-one, l-adic sheaf on $(\mathbf{G}_m)^n$ associated to the exponential sums (1.2), H_c^n denotes l-adic cohomology with proper supports, and F is the Frobenius automorphism. Since $\mathscr{L}_\psi(f)$ is pure of weight 0, it follows from Deligne's theorem [2] that each ρ_i is an algebraic integer pure of weight $w(\rho_i)$ for some integer $w(\rho_i)$, $0 \leqq w(\rho_i) \leqq n$, i.e.,

$$|\rho_i| = \sqrt{q}^{w(\rho_i)}$$

for every complex imbedding of ρ_i. For $0 \leqq k \leqq n$, let w_k be the number of roots of weight k:

$$w_k = \#\{\rho_i | w(\rho_i) = k\} .$$

The purpose of this article is to give an explicit formula for w_k under a certain hypothesis on $\Delta(f)$. We shall also conjecture a general formula for w_k. For any face

σ of $\Delta(f)$ that contains the origin, set

$$v(\sigma) = (-1)^{\dim \sigma} \sum_{(0,\ldots,0)\in\tau\subseteq\sigma} (-1)^{\dim \tau}(\dim \tau)!\, \tilde{V}(\tau)\,, \tag{1.9}$$

where the sum is over all subfaces τ of σ that contain the origin. Let σ_0 be the smallest face of $\Delta(f)$ that contains the origin. We call $\Delta(f)$ *simplicial with respect to the origin* if $\dim \Delta(f) = n$ and σ_0 is contained in exactly $n - \dim \sigma_0$ faces of dimension $n-1$ of $\Delta(f)$. Note that this condition is always satisfied when $n = 2$.

Theorem 1.10. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin. If p is sufficiently large (depending on $\Delta(f)$), then*

$$w_k = \sum_{\substack{(0,\ldots,0)\in\sigma \\ \dim\sigma = k}} v(\sigma)\,, \tag{1.11}$$

the sum being over all faces σ of $\Delta(f)$ that contain the origin and have dimension k.

Example. Let $f(x, y) = x^2y^4 + x^{-3}y^6 + y^8$. Then $\Delta(f)$ is the quadrilateral in $\mathbf{R}^2$ with vertices $(0, 0)$, $(2, 4)$, $(-3, 6)$, and $(0, 8)$. If $p \neq 2, 3$, then f is nondegenerate, so $L(f, t)^{-1}$ is a polynomial of degree $2!\,V(f) = 40$. Besides $\Delta(f)$, there are three other faces containing the origin: $\sigma_0 = (0, 0)$, $\sigma_1 =$ line segment from $(0, 0)$ to $(2, 4)$, and $\sigma_2 =$ line segment from $(0, 0)$ to $(-3, 6)$. We have $\tilde{V}(\sigma_0) = 1$, $\tilde{V}(\sigma_1) = 2$, and $\tilde{V}(\sigma_2) = 3$. The theorem implies that for p sufficiently large (in fact, in this example one needs to exclude only $p = 2, 3$) $w_2 = 40 - 2 - 3 + 1 = 36$, $w_1 = (2-1) + (3-1) = 3$, and $w_0 = 1$. In particular,

$$|S_i(f)| \leqq 36q^i + 3\sqrt{q^i} + 1\,.$$

Remark. If the origin is an interior point of $\Delta(f)$, there are no k-dimensional faces containing the origin when $k < n$. Hence $w_k = 0$ for $k < n$, i.e., the roots of the exponential sum are pure of weight n. This special case of Theorem 1.10 was proved in [1].

The basic idea of the proof is as follows. For $f \in \mathbf{F}_q[x_1, \ldots, x_n]$ one can form an exponential sum on $(\mathbf{G}_m)^n$ or on $\mathbf{A}^n$. In the appendix to [1], we explained how the associated cohomology on $(\mathbf{G}_m)^n$ decomposes and identified one of the "pieces" as being the cohomology on $\mathbf{A}^n$. This allowed us to extend some of our results from exponential sums on $(\mathbf{G}_m)^n$ to exponential sums on $\mathbf{A}^n$. The main observation of this paper is that when $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ and $\Delta(f)$ is simplicial with respect to the origin, there is an analogous decomposition corresponding to the codimension-one faces of $\Delta(f)$ and each of the resulting pieces is pure, i.e., all eigenvalues of Frobenius have the same weight. The number of eigenvalues of a given weight is then the sum of the dimensions of the appropriate pieces, which can be explicitly computed.

To state the general conjecture, we first rewrite formula (1.11). For any face σ of $\Delta(f)$, let $F_\sigma(j)$ be the number of j-dimensional faces of $\Delta(f)$ that contain σ. It is straightforward to check that

$$\sum_{\substack{(0,\ldots,0)\in\sigma \\ \dim\sigma = k}} v(\sigma) = (-1)^k \sum_{\substack{(0,\ldots,0)\in\sigma \\ \dim\sigma \leqq k}} (-1)^{\dim\sigma} F_\sigma(k)(\dim\sigma)!\, \tilde{V}(\sigma)\,. \tag{1.12}$$

Conjecture 1.13. *Let* $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ *be nondegenerate and suppose* $\dim \Delta(f) = n$. *Then*

$$w_k = (-1)^k \sum_{\substack{(0,\ldots,0) \in \sigma \\ \dim \sigma \leqq k}} (-1)^{\dim \sigma} \left(F_\sigma(k) + F_\sigma(k-1) - \binom{n - \dim \sigma}{n-k+1} \right) (\dim \sigma)! \, \tilde{V}(\sigma) , \tag{1.14}$$

the sum being over all faces σ *of* $\Delta(f)$ *that contain the origin and have dimension* $\leqq k$.

Note that when $\Delta(f)$ is simplicial with respect to the origin,

$$F_\sigma(k-1) = \binom{n - \dim \sigma}{n-k+1},$$

so by (1.12), Theorem 1.10 is a special case of Conjecture 1.13. The restriction that $\dim \Delta(f) = n$ is not serious, since, if $\dim \Delta(f) < n$, one can make a coordinate change on $(\mathbf{G}_m)^n$ that transforms f into a Laurent polynomial involving only $\dim \Delta(f)$ variables (coordinate changes on $(\mathbf{G}_m)^n$ preserve nondegeneracy).

Remark added: Our evidence for this conjecture was based on the simplicial case (Theorem (1.10)), the three variable case (which we hope to publish subsequently) and several higher dimensional examples. Denef and Loeser have recently announced a proof of a general formula for the weights w_k which disproves Conjecture 1.13 in general.

Example. In [8], Katz finds a lower bound for the "generic" number of roots of highest weight for exponential sums involving linear polynomials on two types of hypersurfaces. By a well-known procedure, these exponential sums can be expressed as exponential sums on affine space. Restricting to $(\mathbf{G}_m)^n$ does not change the number of roots of highest weight. Theorem 1.10 gives an explicit formula (when p is sufficiently large) for the number of roots of highest weight in one of the cases considered by Katz [8, Corollary 6.5] (this also follows from [1, Theorem 5.17]). Conjecture 1.13 implies that Katz's lower bound in the other case ([8, Theorem 6.9 and Lemmas 6.12 and 6.13] is actually the precise number of roots of highest weight in the generic case. Theorem 1.10 and Conjecture 1.13 also give explicit conditions on the linear polynomial that insure it will be sufficiently generic.

The paper is organized as follows. In section 2, we recall the results from [1] on which this paper is based. In section 3, we describe the decomposition of the Koszul complex corresponding to the faces of $\Delta(f)$ containing the origin. This allows us to reduce the problem of obtaining archimedian estimates to the problem of obtaining p-adic estimates (equation (3.21)). In section 4, we reduce this p-adic question to some combinatorial formulas (Lemma 4.13). Finally, in section 5, we prove the combinatorial formulas.

§2. Preliminaries

We begin by recalling the main results of [1]. Let $\mathbf{Q}_p$ be the field of p-adic numbers and $\mathbf{Z}_p$ the ring of p-adic integers. Set $\Omega_1 = \mathbf{Q}_p(\zeta_p)$, a totally ramified extension of

$\mathbf{Q}_p$ of degree $p-1$. Denote by K the unramified extension of $\mathbf{Q}_p$ of degree a and set $\Omega_0 = K(\zeta_p)$. The Frobenius automorphism $x \to x^p$ of $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ lifts to a generator τ of $\mathrm{Gal}(\Omega_0/\Omega_1)$ ($\cong \mathrm{Gal}(K/\mathbf{Q}_p)$) by requiring $\tau(\zeta_p) = \zeta_p$. Let Ω be the completion of an algebraic closure of Ω_0. Denote by "ord" the additive valuation on Ω normalized by $\mathrm{ord}\ p = 1$, and by "ord_q" the additive valuation normalized by $\mathrm{ord}_q\ q = 1$.

Let $C(f)$ be the cone in $\mathbf{R}^n$ over $\Delta(f)$, i.e., $C(f)$ is the union of all rays in $\mathbf{R}^n$ emanating from the origin and passing through $\Delta(f)$. The lattice points $M(f) = \mathbf{Z}^n \cap C(f)$ in $C(f)$ form an additive monoid. For each $u \in M(f)$, the half-line from the origin through u intersects $\Delta(f)$ in a face that does not contain the origin. Let $\sum_{i=1}^n \varepsilon_i x_i = 1$ be the equation of a hyperplane passing through this face. We define the *weight* of u, denoted $w(u)$, by

$$w(u) = \sum_{i=1}^{n} \varepsilon_i u_i \,. \tag{2.1}$$

Clearly, $w(u)$ is the smallest nonnegative real number such that $w(u)\Delta(f)$ (the dilation of $\Delta(f)$ by the factor $w(u)$) contains u. There is a positive integer M such that w maps $M(f)$ into $M^{-1}\mathbf{Z}_+$.

The weight function is easily seen to have the following properties:

Lemma 2.2 (a) $w(u) = \inf\{\sum_{j\in J} \alpha_j \mid \sum_{j\in J} \alpha_j j = u, \alpha_j \in \mathbf{Q}, \alpha_j \geqq 0\}$. (*This infimum is easily seen to be a minimum.*)

(b) *If* $l \in \mathbf{Z}_+$, *then* $w(lu) = lw(u)$.

(c) $w(u+u') \leqq w(u) + w(u')$, *with equality holding if and only if* u *and* u' *are cofacial, i.e.,* $w(u)^{-1}u$ *and* $w(u')^{-1}u'$ *lie on the same closed face of* $\Delta(f)$.

Let $E(t)$ be the Artin-Hasse exponential series: $E(t) = \exp(\sum_{i=0}^{\infty} t^{p^i}/p^i)$. Let $\gamma \in \Omega_1$ be a solution of $\sum_{i=0}^{\infty} t^{p^i}/p^i = 0$ satisfying $\mathrm{ord}\,\gamma = 1/(p-1)$, and consider

$$\theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Omega_1[[t]] \,. \tag{2.3}$$

The series $\theta(t)$ is a splitting function in Dwork's terminology [4]. In particular, its coefficients satisfy $\mathrm{ord}\,\lambda_i \geqq i/(p-1)$. We fix a choice $\tilde{\gamma}$ of M-th root of γ in Ω and set $\tilde{\Omega}_0 = \Omega_0(\tilde{\gamma})$, $\tilde{\Omega}_1 = \Omega_1(\tilde{\gamma})$. We extend $\tau \in \mathrm{Gal}(\Omega_0/\Omega_1)$ to a generator of $\mathrm{Gal}(\tilde{\Omega}_0/\tilde{\Omega}_1)$ by setting $\tau(\tilde{\gamma}) = \tilde{\gamma}$. Let $\tilde{\mathcal{O}}_0$ be the ring of integers of $\tilde{\Omega}_0$.

Let $\hat{f} = \sum_{j\in J} \hat{a}_j x^j \in K[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be the Teichmüller lifting of f, i.e., $(\hat{a}_j)^q = \hat{a}_j$ and the reduction of $\hat{f}$ mod p is f. Set

$$F(x) = \prod_{j\in J} \theta(\hat{a}_j x^j) \tag{2.4}$$

$$F_0(x) = \prod_{i=0}^{a-1} F^{\tau^i}(x^{p^i}) \,. \tag{2.5}$$

Define an operator ψ on formal Laurent series by

$$\psi\left(\sum_{u\in M(f)} A_u x^u\right) = \sum_{u\in M(f)} A_{pu} x^u \,. \tag{2.6}$$

The operator $\alpha = \psi^a \circ F_0$ is a completely continuous $\tilde{\Omega}_0$-linear endomorphism of the space B, where

$$B = \left\{ \sum_{u \in M(f)} A_u \tilde{\gamma}^{Mw(u)} x^u \mid A_u \in \tilde{\Omega}_0, \quad A_u \to 0 \quad \text{as} \quad u \to \infty \right\}.$$

The Dwork trace formula asserts that

$$S_i(f) = (q^i - 1)^n \operatorname{Tr}(\alpha^i \mid B), \tag{2.7}$$

where the nontrivial additive character Ψ implicit on the left-hand side of (2.7) is given by

$$\Psi(t) = \theta(1)^{\operatorname{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(t)} \tag{2.8}$$

for $t \in \mathbf{F}_q$ (by [4, Lemma 4.1], $\theta(1)$ is a primitive p-th root of unity). Equivalently, one can define an operator δ on formal power series with constant term 1 by $g(t)^\delta = g(t)/g(qt)$, and then (2.7) takes the form

$$L(f, t)^{(-1)^{n-1}} = \det(I - t\alpha)^{\delta^n} \tag{2.9}$$

via the relationship $\det(I - t\alpha) = \exp(-\sum_{i=1}^{\infty} \operatorname{Tr}(\alpha^i) t^i / i)$.

Set $f_i = x_i \partial f / \partial x_i$, $\hat{f}_i = x_i \partial \hat{f} / \partial x_i$. We let $\gamma_l = \sum_{i=0}^{l} \gamma^{p^i} / p^i$, which by the definition of γ satisfies

$$\operatorname{ord} \gamma_l \geqq \frac{p^{l+1}}{p-1} - l - 1 . \tag{2.10}$$

For $i = 1, \ldots, n$, define differential operators $\hat{D}_i$ by

$$\hat{D}_i = E_i + \hat{H}_i , \tag{2.11}$$

where $E_i = x_i \partial / \partial x_i$ and $\hat{H}_i(x) = \sum_{l=0}^{\infty} \gamma_l p^l \hat{f}_i^{\tau^l}(x^{p^l})$. The $\hat{D}_i$ operate on B and one verifies that they commute with one another and that

$$\alpha \circ \hat{D}_i = q \hat{D}_i \circ \alpha \tag{2.12}$$

for $i = 1, \ldots, n$.

Let $K_\cdot(B, \{\hat{D}_i\}_{i=1}^n)$ be the Koszul complex on B formed from $\hat{D}_1, \ldots, \hat{D}_n$, i.e. for $0 \leqq l \leqq n$,

$$K_l(B) = \bigoplus_{|T| = l} B e_T ,$$

where the sum runs over all subsets $T \subseteq S = \{1, 2, \ldots, n\}$ of cardinality l, and if $T = \{i_1, \ldots, i_l\}$ with $i_1 < \ldots < i_l$, then e_T is abbreviated notation for $e_{i_1} \wedge \ldots \wedge e_{i_l}$ (the $\{e_i\}_{i \in S}$ being a set of formal symbols). Define an endomorphism α_l of K_l by

$$\alpha_l = \bigoplus_{|T| = l} q^l \alpha .$$

Equation (2.12) implies that $\alpha_\cdot$ is a chain map on the Koszul complex $K_\cdot$. It then follows from (2.9) that

$$L(f, t)^{(-1)^{n-1}} = \prod_{l=0}^{n} \det(I - t\alpha_l \mid K_l)^{(-1)^l} . \tag{2.13}$$

Passing to homology, we conclude that

$$L(f,t)^{(-1)^{n-1}} = \prod_{l=0}^{n} \det(I - t\bar{\alpha}_l \mid H_l(K.))^{(-1)^l}, \tag{2.14}$$

where $\bar{\alpha}_l$ denotes the endomorphism of $H_l(K.)$ induced by α_l.

Some of the main results of [1] are summarized in the following theorem.

Theorem 2.15. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate. The complex $K.(B, \{\hat{D}_i\}_{i=1}^n)$ is acyclic in dimension $> n - \dim \Delta(f)$. For $0 \leqq l \leqq n - \dim \Delta(f)$, $H_l(K.(B, \{\hat{D}_i\}_{i=1}^n))$ as $\bar{\alpha}_l$-module is isomorphic to the direct sum of $\binom{n - \dim \Delta(f)}{l}$ copies of $H_0(K.(B, \{\hat{D}_i\}_{i=1}^n))$ as $q^l\bar{\alpha}_0$-module. In addition, $H_0(K.(B, \{\hat{D}_i\}_{i=1}^n))$ has dimension $(\dim \Delta(f))!\, \tilde{V}(f)$ and $\bar{\alpha}_0$ is invertible. Hence*

$$L(f,t)^{(-1)^{n-1}} = P(t)^{\delta^{n-\dim \Delta(f)}}, \tag{2.16}$$

where $P(t) = \det(I - t\bar{\alpha}_0 \mid H_0(K.(B, \{\hat{D}_i\}_{i=1}^n))) \in \mathbf{Q}(\zeta_p)[t]$ is a polynomial of degree $(\dim \Delta(f))!\, \tilde{V}(f)$. Furthermore, for p sufficiently large (depending on $\Delta(f)$) the reciprocal roots ρ_i of $P(t)$ are algebraic integers pure of weight $w(\rho_i)$ for some integer $w(\rho_i)$, $0 \leqq w(\rho_i) \leqq \dim \Delta(f)$.

Proof. All statements except the last follow immediately from Theorem 2.9, Corollary 2.19, and Theorem 3.13 of [1]. When $\dim \Delta(f) = n$, the last statement is a consequence of [1, Theorem 4.2] and Deligne's fundamental theorem [2]. If $\dim \Delta(f) < n$, one can make a coordinate change on $(\mathbf{G}_m)^n$ that transforms f into a Laurent polynomial involving $\dim \Delta(f)$ variables and then apply [1, Theorem 4.2].

§3. Decomposition of the Koszul complex

Suppose H is a hyperplane in $\mathbf{R}^n$ passing through the origin: $\sum_{j=1}^n a_j X_j = 0$. Suppose in addition the a_j are rational. We denote by D_H the differential operator on B defined by $D_H = \sum_{j=1}^n a_j \hat{D}_j$.

Suppose that $\Delta(f)$ is simplicial with respect to the origin and let $H_1, \ldots, H_{n - \dim \sigma_0}$ be the hyperplanes through the codimension-one faces of $\Delta(f)$ that contain the origin, where σ_0 is the smallest face containing the origin. Let $H_{n - \dim \sigma_0 + 1}, \ldots, H_n$ be additional hyperplanes through the origin chosen so that the normal vectors to $H_1, \ldots, H_n$ form a basis for $\mathbf{R}^n$. We may assume the equation of each H_i has the form

$$\sum_{j=1}^{n} a_{ij} X_j = 0,$$

where $a_{ij} \in \mathbf{Z}$ for all i, j and $\{a_{ij}\}_{j=1}^n$ have greatest common divisor 1. The differential operators $\{D_{H_i}\}_{i=1}^n$ on B commute with one another and satisfy (2.12).

Lemma 3.1. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $p \nmid \det(a_{ij})$. Then Theorem 2.15 is valid with $\{\hat{D}_i\}_{i=1}^n$ replaced by $\{D_{H_i}\}_{i=1}^n$.*

Proof. Since

$$\left(\sum_{j=1}^{n} a_{ij}x_j\partial f/\partial x_j\right)_\sigma = \sum_{j=1}^{n} a_{ij}x_j\partial f_\sigma/\partial x_j\,,$$

the hypotheses that f be nondegenerate and that $p \nmid \det(a_{ij})$ imply that for all faces σ of $\Delta(f)$ not containing the origin, the polynomials $(\sum_{j=1}^{n} a_{ij}x_j\partial f/\partial x_j)_\sigma$, $i = 1, \ldots, n$, have no common zero in $(\bar{\mathbf{F}}_q^\times)^n$. One can then repeat the arguments of [1] with the $x_i\partial f/\partial x_i$ replaced by the $\sum_{j=1}^{n} a_{ij}x_j\partial f/\partial x_j$.

For $j = 1, \ldots, n - \dim\sigma_0$, we define $\tilde{\Omega}_0$*-endomorphisms* $\theta_j: B \to B$ by

$$\theta_j\left(\sum_{u\in M(f)} A_u\tilde{\gamma}^{Mw(u)}x^u\right) = \sum_{u\in M(f)\cap H_j} A_u\tilde{\gamma}^{Mw(u)}x^u\,. \tag{3.2}$$

Note that θ_j is also a ring homomorphism of B to itself. It is straightforward to check that

$$D_{H_j}(B) \subseteq \ker\theta_j \qquad \text{for } 1 \leqq j \leqq n - \dim\sigma_0 \tag{3.3}$$

$$\theta_j^2 = \theta_j \qquad \text{for } 1 \leqq j \leqq n - \dim\sigma_0 \tag{3.4}$$

$$\theta_j \circ D_{H_i} \circ \theta_j = \theta_j \circ D_{H_i} \qquad \text{for } 1 \leqq i \leqq n,\ 1 \leqq j \leqq n - \dim\sigma_0\,. \tag{3.5}$$

Set $S = \{1, 2, \ldots, n\}$, $S' = \{1, 2, \ldots, n - \dim\sigma_0\}$. For $A \subseteq S'$, let θ^A be the composition of all θ_j for $j \in A$. Put $B^A = \theta^A(B)$. The maps $D_{H_i}^A \overset{\text{def}}{=} \theta^A \circ D_{H_i}$ for $i \in S$ are stable on B^A. Thus we may form the Koszul complex $K.(B^A, \{D_{H_i}^A\}_{i\in S-A})$, which we abbreviate by $K.(A)$. Note that we ignore the differential operators $D_{H_i}^A$ for $i \in A$ since by (3.3) they are identically zero on B^A. For each subset $A' \subseteq S' - A$, we define a subcomplex $K.(A, A')$ of $K.(A)$ by setting for $0 \leqq l \leqq n - |A|$

$$K_l(A, A') = \bigoplus_{\substack{T\subseteq S-A\\ |T|=l}} \left(\bigcap_{j\in A'-T} (\ker\theta_j \mid B^A)\right) e_T\,, \tag{3.6}$$

where the sum runs over all subsets $T \subseteq S - A$ of cardinality l, and if $A' - T = \varnothing$ we take "$\bigcap_{j\in A'-T}(\ker\theta_j \mid B^A)$" to mean "$B^A$". It is straightforward to check that the boundary maps are stable on the submodules, hence they define a subcomplex. Note that $K.(A) = K.(A, \varnothing)$ and that $K.(\varnothing)$ is the complex $K.(B, \{D_{H_i}\}_{i=1}^n)$.

We can define an action of Frobenius on these complexes. Since the θ_j commute with ψ and with multiplication by F_0, we may define an endomorphism

$$\alpha_A = \psi^a \circ \theta^A(F_0): B^A \to B^A \tag{3.7}$$

that satisfies

$$\alpha_A \circ D_{H_i}^A = qD_{H_i}^A \circ \alpha_A \tag{3.8}$$

for $i = 1, \ldots, n$. As in section 2, this induces an endomorphism $\alpha_A: K.(A) \to K.(A)$ by setting

$$\alpha_{A,l} = \bigoplus_{|T|=l} q^l\alpha_A: K_l(A) \to K_l(A)\,. \tag{3.9}$$

It is straightforward to check that α_A induces endomorphisms $\alpha_{A,A'}$ of the subcomplexes $K.(A, A')$ for all $A' \subseteq S' - A$, as well. We denote by $\bar{\alpha}_A$ and $\bar{\alpha}_{A,A'}$ the endomorphisms of the associated homology groups.

We now explain the arithmetic significance of these complexes. For f as in (1.1) and $A \subseteq S'$, put

$$f_A = \sum_{j \in J \cap (\bigcap_{i \in A} H_i)} a_j x^j \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}] . \tag{3.10}$$

We denote the corresponding exponential sums on $(\mathbf{G}_m)^n$ by $S_i(f_A)$ and the corresponding L-function by $L(f_A, t)$.

In section 2 we associated to f a Koszul complex $K.(B)$ with Frobenius operator α satisfying Theorem 2.15 when f is nondegenerate. The same construction can be applied to each f_A, and we denote the resulting Koszul complex and Frobenius operator by $K.(B(f_A))$ and α_{f_A}, respectively. Let $\hat{D}_{A,i}$, $i = 1, \ldots, n$, be the corresponding differential operators. When f is nondegenerate, so are all the f_A, hence Theorem 2.15 holds for the Koszul complex $K.(B(f_A), \{\hat{D}_{A,i}\}_{i=1}^n)$. In addition, if $p \nmid \det(a_{ij})$ the proof of Lemma 3.1 shows that Theorem 2.15 holds for the Koszul complex $K.(B(f_A), \{D_{A,H_i}\}_{i=1}^n)$, where

$$D_{A,H_i} \overset{\text{def}}{=} \sum_{j=1}^{n} a_{ij} \hat{D}_{A,j} .$$

Comparing $K.(B(f_A), \{\hat{D}_{A,i}\}_{i=1}^n)$ with the complex $K.(A)$ constructed in this section, one sees that $K_l(B(f_A))$ and $K_l(A)$ are naturally identified as Banach spaces with action of Frobenius. Under this identification of spaces, the differential operator D_{A,H_i} is identified with $D_{H_i}^A$ for $i = 1, \ldots, n$. Thus $K.(B(f_A), \{D_{A,H_i}\}_{i=1}^n)$ and $K.(B^A, \{D_{H_i}^A\}_{i=1}^n)$ are isomorphic as complexes with action of Frobenius. The hypothesis that $\Delta(f)$ be simplicial with respect to the origin implies that $\dim \Delta(f_A) = n - |A|$ for each $A \subseteq S'$. Keeping in mind that $D_{H_i}^A = 0$ (as operator on B^A) for $i \in A$, Theorem 2.15 applied to f_A gives:

Lemma 3.11. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin and $p \nmid \det(a_{ij})$. For all $A \subseteq S'$, the complex $K.(A)$ is acyclic in dimension > 0, $\dim H_0(K.(A)) = (n - |A|)!\, \tilde{V}(f_A)$, and $\bar{\alpha}_{A,0}$ is invertible. Put*

$$P_A(t) = \det(I - t\bar{\alpha}_{A,0} \mid H_0(K.(A))) .$$

Then $P_A(t)$ is a polynomial of degree $(n - |A|)!\, \tilde{V}(f_A)$ and

$$L(f_A, t)^{(-1)^{n-1}} = P_A(t)^{\delta^{|A|}} . \tag{3.12}$$

Furthermore, for p sufficiently large (depending on $\Delta(f)$) the reciprocal roots ρ_i of $P_A(t)$ are algebraic integers pure of weight $w(\rho_i)$ for some integer $w(\rho_i)$, $0 \leqq w(\rho_i) \leqq n - |A|$.

Corollary 3.13. *With the notation and hypotheses of Lemma* 3.11, $P_A(t) \in \mathbf{Q}(\zeta_p)[t]$.

We now examine the relations between the complexes $K.(A, A')$. Let $A \subseteq S'$, $A' \subseteq S' - A$, and $j \in A'$. The map $\theta_j : B^A \to B^{A \cup \{j\}}$ induces a map θ_j of complexes

with Frobenius action

$$\theta_j : K.(A, A' - \{j\}) \to K.(A \cup \{j\}, A' - \{j\})$$

whose kernel is naturally identified with $K.(A, A')$. By (3.4) this map is surjective. Hence there is a short exact sequence of complexes with Frobenius action

$$0 \to K.(A, A') \to K.(A, A' - \{j\}) \to K.(A \cup \{j\}, A' - \{j\}) \to 0 . \tag{3.14}$$

The associated long exact homology sequence, the acyclicity of $K.(A)$ in dimension > 0, and induction on $|A'|$ (as in [1, Theorem A.7]) show that $K.(A, A')$ is acyclic in dimension > 0 for all $A' \subseteq S' - A$. The associated long exact homology sequence then reduces to the short exact sequence

$$\begin{aligned} 0 \to H_0(K.(A, A')) &\to H_0(K.(A, A' - \{j\})) \\ &\to H_0(K.(A \cup \{j\}, A' - \{j\})) \to 0 . \end{aligned} \tag{3.15}$$

Let $P_{A,A'}(t) = \det(I - t\bar{\alpha}_{A,A',0} \mid H_0(K.(A, A')))$. The short exact sequence (3.15) gives

$$P_{A,A'}(t) = \frac{P_{A,A'-\{j\}}(t)}{P_{A\cup\{j\},A'-\{j\}}(t)} .$$

Induction on $|A'|$ then implies

$$P_{A,A'}(t)^{(-1)^{|A|}} = \prod_C{}' P_C(t)^{(-1)^{|C|}} , \tag{3.16}$$

where $\prod'_C$ denotes a product over all subsets $C \subseteq S'$ such that $A \subseteq C \subseteq A \cup A'$.

We summarize these results:

Lemma 3.17. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin and $p \nmid \det(a_{ij})$. For all $A \subseteq S'$ and $A' \subseteq S' - A$, the complex $K.(A, A')$ is acyclic in dimension > 0. Furthermore,*

$$P_{A,A'}(t)^{(-1)^{|A|}} = \prod_{A \subseteq C \subseteq A \cup A'} P_C(t)^{(-1)^{|C|}} ,$$

hence $P_{A,A'}(t)$ is a polynomial in $\mathbf{Q}(\zeta_p)[t]$ of degree $(-1)^{|A|} \sum_{A \subseteq C \subseteq A \cup A'} (-1)^{|C|}(n - |C|)! \tilde{V}(f_C)$ and, if p is sufficiently large (depending on $\Delta(f)$), all of its reciprocal roots are algebraic integers pure of weights $\leqq n - |A|$.

Corollary 3.18. *With the notation and hypotheses of Lemma* 3.17,

$$P(t) = \prod_{A \subseteq S'} P_{A,S'-A}(t) ,$$

where $P(t) = P_\varnothing(t)$ is as defined in Theorem 2.15.

Proof. By (3.16) we have

$$\begin{aligned}\prod_{A \subseteq S'} P_{A,S'-A}(t) &= \prod_{A \subseteq S'} \prod_{A \subseteq C \subseteq S'} P_C(t)^{(-1)^{|C|+|A|}} \\ &= \prod_{C \subseteq S'} \prod_{A \subseteq C} P_C(t)^{(-1)^{|C|+|A|}} \\ &= \prod_{C \subseteq S'} P_C(t)^{(-1)^{|C|} \sum_{A \subseteq C} (-1)^{|A|}} \\ &= P_{\varnothing}(t)\end{aligned}$$

since $\sum_{A \subseteq C}(-1)^{|A|} = 0$ if $|C| > 0$.

Remark. What Corollary 3.18 reflects, and what one can show with only slightly more effort (using (3.15)), is that

$$H_0(K.(\varnothing)) \cong \bigoplus_{A \subseteq S'} H_0(K.(A, S' - A)) .$$

The main result of this article (Theorem 1.10) will be a consequence of the following:

Theorem 3.19. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin. For p sufficiently large (depending on $\Delta(f)$), the reciprocal roots of $P_{A,S'-A}(t)$ are algebraic integers pure of weight $n - |A|$.*

Remark. Let σ_A be the face of dimension $n - |A|$ of $\Delta(f)$ which lies on $\bigcap_{i \in A} H_i$. Then Lemma 3.17 implies that $P_{A,S'-A}(t)$ is a polynomial of degree $v(\sigma_A)$. Thus Theorem 1.10 is a consequence of Corollary 3.18 and Theorem 3.19.

Proof of Theorem 3.19. Let Λ be the product of the reciprocal roots of $P_{A,S'-A}(t)$ (so $\Lambda \in \mathbf{Q}(\zeta_p)$ by Lemma 3.17). If p is sufficiently large, then by Lemma 3.17

$$|\Lambda| \leqq \sqrt{q}^{(n-|A|)v(\sigma_A)} \tag{3.20}$$

for each archimedian absolute value on $\mathbf{Q}(\zeta_p)$. It is well known that at all primes $l \neq p$, Λ has trivial valuation. Hence if we can show

$$|\Lambda|_p \leqq \sqrt{q}^{-(n-|A|)v(\sigma_A)} \tag{3.21}$$

for every normalized nonarchimedian absolute value on $\mathbf{Q}(\zeta_p)$ lying over p, then the usual product formula for $\mathbf{Q}(\zeta_p)$ will imply that

$$|\Lambda| = \frac{1}{|\Lambda|_p} = \sqrt{q}^{(n-|A|)v(\sigma_A)} .$$

Thus each reciprocal root of $P_{A,S'-A}(t)$ must have weight exactly $n - |A|$.

The proof of (3.21) will be the object of the next two sections.

§4. *p*-Adic estimates

The Newton polyhedron can be used to define a filtration on certain rings. Let

$$R = \mathbf{F}_q[x^u \mid u \in M(f)] . \tag{4.1}$$

The weight function w defines an increasing filtration on R by setting for $i \in \mathbf{Z}_+$

$$R_{i/M} = \left\{ \sum_u b_u x^u \in R \mid w(u) \leqq i/M \text{ for all } u \text{ with } b_u \neq 0 \right\}. \tag{4.2}$$

When $A \subseteq S'$, this induces in the obvious manner filtrations on the subrings

$$R_A = \left\{ \sum_u b_u x^u \in R \mid b_u = 0 \text{ if } u \notin \bigcap_{i \in A} H_i \right\}. \tag{4.3}$$

Associated to each of these filtered rings is a graded ring $\bar{R}_A = \oplus_{i \in \mathbf{Z}} \bar{R}_A^{(i/M)}$, where $\bar{R}_A^{(i/M)} = R_{A,i/M}/R_{A,(i-1)/M}$. The rings R_A and $\bar{R}_A$ are identical as $\mathbf{F}_q$-vector spaces, but multiplication in $\bar{R}_A$ obeys the rule

$$x^u x^{u'} = \begin{cases} x^{u+u'} & \text{if } u \text{ and } u' \text{ are cofacial} \\ 0 & \text{otherwise} . \end{cases} \tag{4.4}$$

Let $f_{H_i} = \sum_{j=1}^n a_{ij} x_j \partial f/\partial x_j \in R_1$ and let $f_{H_i}^A = \theta^A \circ f_{H_i} \in R_{A,1}$. Note that $f_{H_i}^A = 0$ if $i \in A$. Denote by $\bar{f}_{H_i}$ (resp. $\bar{f}_{H_i}^A$) the image of f_{H_i} (resp. $f_{H_i}^A$) in $\bar{R}$ (resp. $\bar{R}_A$).

Lemma 4.5. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin and $p \nmid \det(a_{ij})$. For any subset $A \subseteq S'$, the set $\{\bar{f}_{H_i}^A\}_{i \notin A}$ taken in any order forms a regular sequence on $\bar{R}_A$, hence the Koszul complex $K.(\bar{R}_A, \{\bar{f}_{H_i}^A\}_{i \notin A})$ is acyclic in positive dimension. Furthermore,*

$$\dim_{\mathbf{F}_q} H_0(K.(\bar{R}_A, \{\bar{f}_{H_i}^A\}_{i \notin A})) = (n - |A|)!\, \tilde{V}(\sigma_A) ,$$

where $\sigma_A = \Delta(f) \cap (\bigcap_{i \in A} H_i)$.

Proof. The hypotheses imply that f_A is also nondegenerate and that $\Delta(f_A)$ is also simplicial with respect to the origin. Since $f_{H_i}^A = (f_A)_{H_i}$, the argument of Lemma 3.1 shows that one can simply repeat the proof of [1, Theorem 2.14].

One defines $\theta_j : \bar{R}_A \to \bar{R}_{A \cup \{j\}}$ as in (3.2). For $A' \subseteq S' - A$, this allows us to define, as in (3.6), subcomplexes $\bar{K}.(A, A')$ of the Koszul complex $\bar{K}.(A) \overset{\text{def}}{=} K.(\bar{R}_A, \{\bar{f}_{H_i}^A\}_{i \notin A})$. Proceeding as in the deduction of Lemma 3.17 from Lemma 3.11, we deduce:

Lemma 4.6. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin and $p \nmid \det(a_{ij})$. For all $A \subseteq S'$ and $A' \subseteq S' - A$, the complex $\bar{K}.(A, A')$ is acyclic in dimension > 0. Furthermore,*

$$\dim_{\mathbf{F}_q} H_0(\bar{K}.(A, A')) = (-1)^{|A|} \sum_{A \subseteq C \subseteq A \cup A'} (-1)^{|C|} (n - |C|)!\, \tilde{V}(\sigma_C) .$$

In particular,

$$\dim_{\mathbf{F}_q} H_0(\bar{K}.(A, S' - A)) = \nu(\sigma_A) .$$

Set $\bar{R}_{A,A'} = \bigcap_{j \in A'} (\ker \theta_j \mid \bar{R}_A)$. By Lemma 4.6, $H_0(\bar{K}.(A, A')) = \bar{R}_{A,A'}/\sum_{i \notin A} \bar{f}_{H_i}^A \bar{R}_{A,A'-\{i\}}$ is a finite-dimensional graded ring. Let $a_i(A, A')$ be the dimension over $\mathbf{F}_q$ of the graded piece of degree i/M of

$H_0(\bar{K}.(A, A'))$. It can be shown (as in [9, Lemma 2.9] and [3, Lemma 1.5]) that $a_i(A, A') = 0$ for $i > (n - |A|)M$.

Lemma 4.7. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin and $p \nmid \det(a_{ij})$. The Newton polygon of the polynomial $P_{A,A'}(t)$ with respect to the valuation ord_q lies above the Newton polygon of the polynomial $\prod_{i=0}^{(n-|A|)M}(1 - q^{i/M}t)^{a_i(A,A')}$ and their endpoints coincide.*

Proof. This is proved by the method used in [1, Theorem 3.10]. The coincidence of the endpoints, which was not proved in [1], follows as in [5, paragraph preceding Theorem 7.1].

It follows that the coefficient of the highest degree term of $P_{A,A'}(t)$ has ordinal (with respect to the valuation ord_q) equal to $\sum_{i=0}^{(n-|A|)M} a_i(A, A')\dfrac{i}{M}$. We do not know how to evaluate this explicitly in general, since we do not have a formula for $a_i(A, A')$. The main point of this section is that we can evaluate it in the case where $A' = S - A$, which will be sufficient to establish (3.21).

Theorem 4.8. *Let $f \in \mathbf{F}_q[x_1, \ldots, x_n, (x_1 \ldots x_n)^{-1}]$ be nondegenerate and suppose $\Delta(f)$ is simplicial with respect to the origin and $p \nmid \det(a_{ij})$. Then*

$$a_i(A, S' - A) = a_{(n-|A|)M-i}(A, S' - A)\,.$$

Since $\sum_{i=0}^{(n-|A|)M} a_i(A, S' - A) = \deg P_{A,S'-A}(t) = \nu(\sigma_A)$, we have as an immediate consequence

$$\sum_{i=0}^{(n-|A|)M} a_i(A, S' - A)\frac{i}{M} = \frac{n - |A|}{2}\nu(\sigma_A)\,, \tag{4.9}$$

which establishes (3.21).

Proof of Theorem 4.8. Fix $A \subseteq S'$ and consider the complex $\bar{K}.(A, S' - A)$. Since this complex is acyclic in dimension > 0, we have for $i = 0, 1, \ldots, nM$

$$a_i(A, S' - A) = \sum_{l=0}^{n-|A|}(-1)^l \dim_{\mathbf{F}_q} \bar{K}_l(A, S' - A)^{(i/M-l)}\,, \tag{4.10}$$

where $\bar{K}_l(A, S' - A)^{(i/M-l)}$ denotes the homogeneous part of degree $i/M - l$ of $\bar{K}_l(A, S' - A)$. From its definition, we see that $\bar{K}_l(A, S' - A)$ (as $\mathbf{F}_q$-vector space) is identified with $\oplus_{\substack{T \subseteq S-A \\ |T|=l}}(\bigcap_{j \in S'-(A \cup T)}(\ker \theta_j \mid \bar{R}_A))$. Furthermore, $\bigcap_{j \in S'-(A \cup T)}(\ker \theta_j \mid \bar{R}_A)$ as $\mathbf{F}_q$-vector space has basis equal to the set of all monomials x^u such that $u \in (M(f) \cap (\bigcap_{j \in A} H_j)) - (\bigcup_{j \in S'-(A \cup T)} H_j)$. Let $W_T(l)$ be the number of such monomials having weight l/M. Then

$$\dim_{\mathbf{F}_q} \bar{K}_l(A, S' - A)^{(i/M-l)} = \sum_{\substack{T \subseteq S-A \\ |T|=l}} W_T(i - lM)\,, \tag{4.11}$$

hence

$$a_i(A, S' - A) = \sum_{l=0}^{n-|A|}(-1)^l \sum_{\substack{T \subseteq S-A \\ |T|=l}} W_T(i - lM)\,. \tag{4.12}$$

Note that this sum may be truncated at $l =$ greatest integer in i/M, since $W_T(l) = 0$ for $l < 0$.

The desired result is trivial when $A = S$, so we assume $A \neq S$. The main lemma required for the proof of Theorem 4.8 is the following.

Lemma 4.13. *Suppose that $T \subseteq S - A \neq \emptyset$ and r is an integer, $0 \leqq r \leqq M - 1$.*

(a) *There exists a polynomial $p_{T,r}(x) \in \mathbf{Q}[x]$ of degree $n - |A| - 1$ such that for $l > 0$, $l \equiv r \pmod{M}$ we have $W_T(l) = p_{T,r}(l)$.*

(b) *If r' denotes the unique integer, $0 \leqq r' \leqq M - 1$, such that $r + r' \equiv 0 \pmod{M}$, then $p_{S-(A \cup T),r'}(x) = (-1)^{n-|A|-1} p_{T,r}(-x)$.*

(c) *One has*

$$p_{T,0}(0) = \begin{cases} 0 & \text{if } T \cap S' \neq \emptyset \text{ and } A \cup (T \cap S') \neq S' \\ 1 & \text{if } T \cap S' \neq \emptyset \text{ and } A \cup (T \cap S') = S' \\ (-1)^{n-|A|-1} & \text{if } T \cap S' = \emptyset \text{ and } A \neq S' \\ 1 + (-1)^{n-|S'|-1} & \text{if } T \cap S' = \emptyset \text{ and } A = S' \end{cases} .$$

We postpone the proof of Lemma 4.13 until the next section and continue with the proof of Theorem 4.8. Since $a_i(A, S' - A) = 0$ for $i > (n - |A|)M$, equation (4.12) and Lemma 4.13(a) imply that

$$\sum_{l=0}^{n-|A|} (-1)^l \sum_{\substack{T \subseteq S - A \\ |T| = l}} p_{T,r}(x - lM) = 0 . \tag{4.14}$$

We consider two cases, according as to whether i is or is not divisible by M. First fix r, $1 \leqq r \leqq M - 1$, and let $i \equiv r \pmod{M}$. Let s be the greatest integer in i/M. Equation (4.14) implies

$$\sum_{l=0}^{s} (-1)^l \sum_{\substack{T \subseteq S - A \\ |T| = l}} p_{T,r}(x - lM) = \sum_{l=s+1}^{n-|A|} (-1)^{l+1} \sum_{\substack{T \subseteq S - A \\ |T| = l}} p_{T,r}(x - lM) . \tag{4.15}$$

Applying Lemma 4.13(b) to the right-hand side of (4.15) gives

$$\begin{aligned} &\sum_{l=0}^{s} (-1)^l \sum_{\substack{T \subseteq S - A \\ |T| = l}} p_{T,r}(x - lM) \\ &= \sum_{l=s+1}^{n-|A|} (-1)^{l+1} \sum_{\substack{T \subseteq S - A \\ |T| = n - |A| - l}} (-1)^{n-|A|-1} p_{T,r'}(lM - x) \\ &= \sum_{l=0}^{n-|A|-s-1} (-1)^l \sum_{\substack{T \subseteq S - A \\ |T| = l}} p_{T,r}((n - |A|)M - x - lM) . \end{aligned} \tag{4.16}$$

Evaluating at $x = i$ and using (4.12) and Lemma 4.13(a) shows immediately that $a_i(A, S' - A) = a_{(n-|A|)M-i}(A, S' - A)$.

Now consider the case $r = 0$ so that $i \equiv 0 \pmod{M}$ and $s = i/M$. We would like to evaluate (4.16) at $x = i$ as before, but one must be slightly careful because Lemma 4.13(a) need not remain valid when $l = 0$. Suppose first that $i \neq 0$,

$(n-|A|)M$. One has by (4.12) and Lemma 4.13(a) that

$$a_i(A, S'-A) = \sum_{l=0}^{s-1} (-1)^l \sum_{\substack{T \subseteq S-A \\ |T|=l}} p_{T,r}(i-lM) + (-1)^s \sum_{\substack{T \subseteq S-A \\ |T|=s}} W_T(0)$$

$$a_{(n-|A|)M-i}(A, S'-A) = \sum_{l=0}^{n-|A|-s-1} (-1)^l \sum_{\substack{T \subseteq S-A \\ |T|=l}} p_{T,r'}((n-|A|)M-i-lM) +$$

$$(-1)^{n-|A|-s} \sum_{\substack{T \subseteq S-A \\ |T|=n-|A|-s}} W_T(0) .$$

The result then follows from (4.16) evaluated at $x = i$, Lemma 4.13(c), and the observation that for $T \subseteq S - A \neq \emptyset$,

$$W_T(0) = \begin{cases} 0 & \text{if } S' \neq A \cup T' \\ 1 & \text{if } S' = A \cup T' . \end{cases} \tag{4.17}$$

From (4.12) and (4.17) one has

$$a_0(A, S'-A) = \begin{cases} 0 & \text{if } S' \neq A \\ 1 & \text{if } S' = A . \end{cases} \tag{4.18}$$

Evaluating (4.12) at $i = (n-|A|)M$ and (4.14) at $x = (n-|A|)M$ and using Lemma 4.13(a) gives

$$a_{(n-|A|)M}(A, S'-A) = (-1)^{n-|A|}(W_{S-A}(0) - p_{S-A,0}(0)) .$$

The desired equality now follows from (4.17) and Lemma 4.13(c).

§5. Proof of Lemma 4.13

To prove Lemma 4.13 we introduce some auxiliary functions. For $T \subseteq S - A$, set $T' = T \cap S'$. Define $\tilde{W}_T(l)$ to be the number of monomials x^u of weight l/M such that $u \in M(f) \cap (\bigcap_{j \in A \cup T'} H_j)$ and let $\hat{W}_T(l)$ be the number of monomials x^u of weight l/M such that $u \in (M(f) \cap (\bigcap_{j \in A \cup T'} H_j)) - \bigcup_{j \in S'-(A \cup T')} H_j$. We have the following relationships.

Lemma 5.1. *For $T \subseteq S - A \neq \emptyset$,*

$$W_T(l) = \sum_{B \subseteq T'} \hat{W}_B(l)$$

$$W_T(l) = \sum_{B \subseteq S'-(A \cup T')} (-1)^{|B|} \tilde{W}_B(l) .$$

Proof. For each $u \in (M(f) \cap (\bigcap_{j \in A} H_j)) - \bigcup_{j \in S'-(A \cup T)} H_j$, there is a unique subset $B \subseteq T'$ such that $u \in (M(f) \cap (\bigcap_{j \in A \cup B} H_j)) - \bigcup_{j \in S'-(A \cup B)} H_j$. This proves the first identity. The second identity is proved by a standard inclusion-exclusion argument.

These auxiliary functions satisfy the following.

Lemma 5.2. *Suppose that* $T \subseteq S - A \neq \varnothing$ *and* r *is an integer,* $0 \leqq r \leqq M-1$. *There exist polynomials* $\tilde{p}_{T,r}(x), \mathring{p}_{T,r}(x) \in \mathbf{Q}[x]$, *both of degree* $n - |A \cup T'| - 1$ *such that*:

(a) *For* $l > 0$, $l \equiv r \pmod{M}$, *we have* $\tilde{W}_T(l) = \tilde{p}_{T,r}(l)$ *and* $\mathring{W}_T(l) = \mathring{p}_{T,r}(l)$.

(b) *If* r' *denotes the unique integer,* $0 \leqq r' \leqq M-1$, *such that* $r + r' \equiv 0 \pmod{M}$, *then* $\mathring{p}_{T,r'}(x) = (-1)^{n-|A\cup T'|-1}\tilde{p}_{T,r}(-x)$.

(c) *One has*

$$\tilde{p}_{T,0}(0) = \begin{cases} 1 & \text{if } A \cup T' \neq S' \\ 0 & \text{if } A \cup T' = S' \text{ and } n - |S'| \text{ is even} \\ 2 & \text{if } A \cup T' = S' \text{ and } n - |S'| \text{ is odd} \end{cases}$$

$$\mathring{p}_{T,0}(0) = \begin{cases} (-1)^{n-|A\cup T'|-1} & \text{if } A \cup T' \neq S' \\ 0 & \text{if } A \cup T' = S' \text{ and } n - |S'| \text{ is even} \\ 2 & \text{if } A \cup T' = S' \text{ and } n - |S'| \text{ is odd} . \end{cases}$$

Remark. Note that Lemma 4.13 now follows in a straightforward manner by substituting the results of Lemma 5.2 into the formulas of Lemma 5.1.

Proof of Lemma 5.2. Suppose first $A \cup T' = S$. Then $\tilde{W}_T(l) = \mathring{W}_T(l) = 0$ for $l > 0$, so the equalities in (a), (b), and (c) are valid with $\tilde{p}_{T,r}$ and $\mathring{p}_{T,r}$ both taken to be the zero polynomial. So the lemma is true if one assigns the degree -1 to the zero polynomial.

Now suppose $A \cup T' \neq S$. Let H_B be the linear variety $\bigcap_{j\in B} H_j$, let $\Delta_B = \Delta(f) \cap H_B$, and let $(l/M)\Delta_B$ denote the dilation of Δ_B by the factor l/M. Let $j_B(l)$ be the number of lattice points in $(l/M)\Delta_B$, $i_B(l)$ the number of lattice points in the interior of $(l/M)\Delta_B$, and $h_B(l)$ the number of lattice points of the boundary of $(l/M)\Delta_B$. It is straightforward to check that for $l > 0$,

$$\tilde{W}_T(l) = h_{A\cup T'}(l) - \sum_{A\cup T' \subset B \subseteq S'} i_B(l) \tag{5.3}$$

$$\mathring{W}_T(l) = h_{A\cup T'}(l) + \sum_{A\cup T' \subset B \subseteq S'} (-1)^{|B|-|A\cup T'|} j_B(l) , \tag{5.4}$$

where we use the notation $A \cup T' \subset B$ to mean that $A \cup T'$ is a proper subset of B, i.e., $A \cup T' \neq B$, and where we make the convention that $j_S(l) = i_S(l) = 1$.

Ehrhart [6] proves that there exist polynomials $i_{B,r}(x), j_{B,r}(x) \in \mathbf{Q}[x]$ of degree $n - |B|$ and $h_r(x) \in \mathbf{Q}[x]$ of degree $n - |A \cup T'| - 1$ such that for $l > 0$, $l \equiv r \pmod{M}$,

$$i_B(l) = i_{B,r}(l)$$

$$j_B(l) = j_{B,r}(l)$$

$$h_{A\cup T'}(l) = h_r(l) .$$

In view of (5.3) and (5.4), this implies the existence of polynomials $\tilde{p}_{T,r}(x)$, $\mathring{p}_{T,r}(x) \in \mathbf{Q}[x]$ satisfying Lemma 5.2(a). In addition, Ehrhart ([6, Corollaire A.2 of Chapitre 6], [7]) shows that

$$i_{B,r'}(x) = (-1)^{n-|B|} j_{B,r}(-x) . \tag{5.5}$$

Since $h_r(x) = j_{A\cup T',r}(x) - i_{A\cup T',r}(x)$, we have

$$h_{r'}(x) = (-1)^{n-|A\cup T'|-1} h_r(-x)\,. \tag{5.6}$$

Starting with (5.4) and applying (5.5), (5.6), and (5.3) we get

$$\begin{aligned}
\mathring{p}_{T,r'}(l) &= h_{r'}(l) + \sum_{A\cup T'\subset B\subseteq S'} (-1)^{|B|-|A\cup T'|} j_{B,r'}(l) \\
&= (-1)^{n-|A\cup T'|-1} h_r(-l) + \sum_{A\cup T'\subset B\subseteq S'} (-1)^{|B|-|A\cup T'|}(-1)^{n-|B|} i_{B,r}(-l) \\
&= (-1)^{n-|A\cup T'|-1} h_r(-l) - (-1)^{n-|A\cup T'|-1} \sum_{A\cup T'\subset B\subseteq S'} i_{B,r}(-l) \\
&= (-1)^{n-|A\cup T'|-1} \tilde{p}_{T,r}(-l)\,,
\end{aligned}$$

which establishes Lemma 5.2(b).

Finally, Ehrhart [6, Remarque 2 following Corollaires 4.2] calculates that $j_{B,0}(0) = 1$ for all B, so (5.5) implies $i_{B,0}(0) = (-1)^{n-|B|}$ and

$$h_0(0) = \begin{cases} 0 & \text{if } n - |A\cup T'| \text{ is even} \\ 2 & \text{if } n - |A\cup T'| \text{ is odd}\,. \end{cases}$$

Lemma 5.2(c) now follows from (5.3) and (5.4).

References

1. Adolphson, A., Sperber, S.: Exponential sums and Newton polyhedra: Cohomology and estimates. Ann. Math. **130**, 367–406 (1989)
2. Deligne, P.: La conjecture de Weil, II. Publ. Math., Inst. Hautes Etud. Sci. **52**, 137–252 (1980)
3. Danilov, V.I.: The geometry of toric varieties. Russ. Math. Surv. **33**(2), 97–154 (1979)
4. Dwork, B.: On the zeta function of a hypersurface. Publ. Math., Inst. Hautes Etud. Sci. **12**, 5–68 (1962)
5. Dwork, B.: On the zeta function of a hypersurface, II. Ann. Math. **80**, 227–299 (1964)
6. Ehrhart, E.: Sur un problème de géométrie diophantienne linéáire, I: Polyèdres et réseaux. J. Reine Angew. Math. **226**, 1–29 (1967)
7. Ehrhart, E.: Démonstration de la loi de réciprocité du polyèdre rationnel. C.R. Acad. Sci. Paris Sér. A–B **265**, A91–A94 (1967)
8. Katz, N.: Perversity and exponential sums. Tohoku Math. J. (to appear)
9. Kouchnirenko, A.G.: Polyèdres de Newton et nombres de Milnor. Invent. Math. **32**, 1–31 (1976)

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Real components of algebraic varieties and étale cohomology

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Let X be a smooth geometrically connected curve over the field $\mathbf{R}$ of real numbers. We denote by S the set of connected components of $X(\mathbf{R})$ for the Euclidean topology and by s its cardinality. Let $\mathbf{R}(X)$ be the function field of X. In 1934, E. Witt proved that the subgroup of the Brauer group of the field $\mathbf{R}(X)$ consisting of classes which are unramified on X is a finite group isomorphic to $(\mathbf{Z}/2)^S$ ([28], Satz III′, see also [12]). For a curve X over $\mathbf{R}$, the Brauer group of $\mathbf{R}(X)$ is 2-torsion and hence coincides with the étale cohomology group $H^2(\mathbf{R}(X), \mathbf{Z}/2)$. Thus Witt's theorem may be rephrased as saying that the unramified subgroup of the étale cohomology group $H^2(\mathbf{R}(X), \mathbf{Z}/2)$ is isomorphic to $(\mathbf{Z}/2)^S$. (We refer the reader to §1 for the definition of the unramified cohomology classes.)

Thus, the following theorem, which is the main result of this paper, appears as a natural extension of Witt's theorem to higher dimensional varieties.

Main theorem. *Let $X/\mathbf{R}$ be a smooth quasi-projective geometrically connected variety over $\mathbf{R}$, of dimension d. Let S denote the (finite) set of connected components of the topological space $X(\mathbf{R})$. Then for any $n \geqq d+1$, the subgroup of the étale cohomology group $H^n(\mathbf{R}(X), \mathbf{Z}/2)$ consisting of classes unramified on X is isomorphic to $(\mathbf{Z}/2)^S$.*

In this theorem, we do not demand that X be projective or that $X(\mathbf{R})$ be compact.

If X is smooth and projective over $\mathbf{R}$, it is a known fact that the number of connected components of the topological space $X(\mathbf{R})$ is a birational invariant ([11], Theorem 13.3). The above theorem shows that this invariant, a priori defined via topology, coincides with the $\mathbf{Z}/2$-rank of the unramified cohomology subgroup of $H^{d+1}(\mathbf{R}(X), \mathbf{Z}/2)$, a birational invariant of a purely algebraic nature.

Another consequence of the theorem is the finiteness of the unramified subgroup of $H^n(\mathbf{R}(X), \mathbf{Z}/2)$ for $n \geqq d+1$. This finiteness has already been used by the second author [20] in proving the finite generation of the Witt group of some affine threefolds. (In the range $3 \leqq n \leqq d$, we do not know whether the unramified subgroup is finite.)

To prove the theorem, we first construct a map h_n from the unramified subgroup of $H^n(\mathbf{R}(X), \mathbf{Z}/2)$ to the group of continuous functions on $X(\mathbf{R})$ with values in $\mathbf{Z}/2$ (see §2.1). We call the map h_n the mod 2 *signature map* (of level n). Its definition uses the Bloch/Ogus results [3] on étale analogues of the Gersten conjecture in algebraic K-theory (as established by Quillen). The main ingredients to prove that, for $n \geqq d+1$, the map h_n is an isomorphism are:

- The results on quadratic forms over fields, beginning with the work of Pfister [21] and further developed by Arason, Elman, Jacob, Lam. The survey [1] will be our basic reference.
- Mahé's result [17] on the separation of real connected components by means of signatures of quadratic spaces.

For the convenience of the reader, the relevant material is reviewed in §1.

The results of [1] are enough to prove injectivity of h_n ($n \geqq d+1$). The proof of surjectivity is more delicate and requires a combination of [1] and [17] (see §2.3). Once connected components are replaced by semi-algebraic components (as defined by Knebusch, Delfs, Coste/Roy) the theorem makes sense over an arbitrary real closed field R, and we actually prove it in this more general context. (For curves, this extension of Witt's theorem is due to Knebusch ([15], p. 56).) Our proof relies on the notion and properties of the real spectrum of a ring, as developed by Coste and Roy ([4], chap. 7). This is not only a matter of elegance: even for the usual real field $\mathbf{R}$ as a ground field, it would be cumbersome to avoid using this notion.

Nevertheless, in the special case where $R = \mathbf{R}$ and moreover $X(\mathbf{R})$ is *compact*, one may, as in [5], use the Stone–Weierstraß approximation theorem and show that every element of the unramified subgroup of $H^n(\mathbf{R}(X), \mathbf{Z}/2)$, $n \geqq d+1$, is a cup-product $\{f, -1, \ldots, -1\}$, where $f \in \mathbf{R}(X)$ is invertible on $X(\mathbf{R})$ (cf. §2.4). In §3, we use these explicit representatives to prove that the Bloch/Ogus spectral sequence [3] *degenerates* for smooth compact real surfaces.

We give two applications of this degeneration. First, if the smooth compact real surface X satisfies a simple geometric condition (which is always fulfilled if X is affine), the degeneration implies that (the 2-torsion subgroup of) the Brauer group of X separates the real connected components of $X(\mathbf{R})$: the map h_2 is surjective (Proposition 3.1.2).

Second, the degeneration result together with some K-theoretic facts leads to lower bounds for the $\mathbf{Z}/2$-rank of the étale cohomology groups $H^n_{\text{ét}}(X, \mathbf{Z}/2)$ of a smooth compact real surface X in terms of the number s of connected components of $X(\mathbf{R})$, such as:

$$\operatorname{rank} H^3_{\text{ét}}(X, \mathbf{Z}/2) \geqq 2s - 1 .$$

These estimates are refinements of those provided by Smith theory.

For a real rational surface X, we can go further and compute the exact rank of $H^3_{\text{ét}}(X, \mathbf{Z}/2)$. This gives a new approach to the known Galois structure ([14], [26], [27]) of the Picard group of such surfaces.

§1. The background

1.1. *The real spectrum and a theorem of Mahé*

For a commutative ring A, the real spectrum $\mathrm{Spec}_r A$ is the set of pairs $(p, <)$ where p is a prime ideal and $<$ denotes an order on the residue class field $\kappa(p)$ of the local ring A_p. Thus, given a point $(p, <)$ of $\mathrm{Spec}_r A$, there is an associated homomorphism $A \to R^p$, where R^p denotes the real closure of the ordered field $(\kappa(p), <)$.

There is a natural topology on $\mathrm{Spec}_r A$ (cf. [4], Chap. 7). If F is a field, $\mathrm{Spec}_r F$ coincides with the space of orderings of F, equipped with the Harrison topology, used in [1].

Assume that A is of finite type over a real closed field R, and let $X = \mathrm{Spec}\, A$ be the usual spectrum, which is a variety of finite type over R. One may then define the semi-algebraic components of the algebraic set $X(R)$ of R-points of X ([4], 2.4 and 7.5). There is an obvious inclusion $X(R) \subset \mathrm{Spec}_r A$. Restriction defines a bijection between the connected components of the topological space $\mathrm{Spec}_r A$ and the semi-algebraic connected components of $X(R)$ ([4], 7.5.1). When R is the real field $\mathbf{R}$, these semi-algebraic connected components coincide with the connected components of $X(\mathbf{R})$ for the Euclidean topology ([4], 2.4.5).

Assume that A is of finite type over R, that it is integral and that $X = \mathrm{Spec}\, A$ is smooth over R. Let $R(X)$ be the field of fractions of A. Then the image of $\mathrm{Spec}_r R(X)$ under the natural inclusion $\mathrm{Spec}_r R(X) \subsetneq \mathrm{Spec}_r A$ is dense in $\mathrm{Spec}_r A$ ([4], 7.6.1).

Let $X = \mathrm{Spec}\, A$ be an affine scheme, and let $W(A)$ be the Witt ring of quadratic spaces over X. We recall that a quadratic space is a projective A-module E of finite rank together with a symmetrical isomorphism of E with its dual.

There exists a signature map from the Witt ring of A to the ring of $\mathbf{Z}$-valued continuous functions of $\mathrm{Spec}_r A$:

$$\mathrm{sgn}: W(A) \to \mathscr{C}(\mathrm{Spec}_r A, \mathbf{Z}),$$

which generalizes the signature map for fields ([1]).

Theorem 1.1.1 (Mahé [17]). *The cokernel of the signature map is a 2-primary torsion group. More precisely, if* $\mathrm{Spec}_r A = U_1 \cup U_2$ *is a decomposition as a union of two disjoint open sets, there exists an integer N and a quadratic space E over X whose signature over each point of* U_1, *resp.* U_2, *is* 2^N, *resp* 0. □

1.2. *Etale cohomology*

Throughout this paper, by cohomology, we mean étale cohomology (dropping the subscript "*ét*" in most places), the main exception being the Zariski cohomology of the Bloch–Ogus sheaves (see below).

If F is a field, $H^n(F, \cdot)$ will denote étale cohomology of $\mathrm{Spec}\, F$, i.e. Galois cohomology of the absolute Galois group $\mathrm{Gal}(F_{sep}/F)$.

We shall often be concerned with the group $H^n_{ét}(F, \mathbf{Z}/2)(\mathrm{char}(F) \neq 2)$, which we shall abbreviate to $H^n(F)$. Given elements a_i $(i = 1, \ldots, n)$ in the multiplicative

group F^* of the field F, we shall write $\{a_1, \ldots, a_n\}$ for the cup-product of the classes of the a_i in $F^*/F^{*2} \simeq H^1(F, \mathbf{Z}/2)$.

Similar conventions will hold over a scheme X in which 2 is invertible ($H^n(X) = H^n_{\acute{e}t}(X, \mathbf{Z}/2)$ etc.)

The following proposition is an easy consequence of a theorem of Serre's [24]. It was first noticed by J. Ax and its proof was brought to our attention by B. Kahn.

Proposition 1.2.1. *Let X be an integral variety over a real closed field R, and let d be the dimension of X. Assume $X(R) = \phi$. Then the cohomological dimension of the function field $R(X)$ is equal to d.*

Proof. Let G be the absolute Galois group of $R(X)$, and let $H \subset G$ be the absolute Galois group of $C(X)$, where $C = R(\sqrt{-1})$. It is known that the cohomological dimension of H is equal to d. Since $C(X)/R(X)$ is of degree at most 2, this implies $cd_p G = d$ for $p \neq 2$. By Artin's results, the hypothesis $X(R) = \phi$ implies that the field $R(X)$ cannot be ordered. Thus, by the Artin–Schreier theorem, the group G does not contain any element of order 2. According to a theorem of Serre [24], this last fact implies $cd_2 G = cd_2 H$, hence $cd_2 G = d$. □

Remark 1.2.2. For X as in the proposition, one may give another proof of $H^n(R(X), \mathbf{Z}/2) = 0$ for $n > d$ by combining Proposition 2.4 and Theorem 1.11 of [1].

Remark 1.2.3. The above proposition proves the main theorem in the case where $X(\mathbf{R}) = \varnothing$.

If R is a real closed field, it is easily shown that $H^n(R) = H^n_{\acute{e}t}(R, \mathbf{Z}/2)$ is isomorphic to $\mathbf{Z}/2$, the non-zero element being the n-fold cup product $\{-1, \ldots, -1\}$. Let K be any field. If $\alpha \in H^n(K)$ and $p \in \mathrm{Spec}_r K$, the image α_p of α under the natural map $H^n(K) \to H^n(R^p)$ is denoted by $hg_n(\alpha)$ ([1]). The following theorem is easily deduced from ([1], Proposition 2.4), together with Proposition 1.2.1 above (cf. [1], proof of Corollary 2.8).

Theorem 1.2.4. *Let $R(X)$ be a function field of transcendence degree d over a real closed field R. For $n > d$, the map hg_n: $H^n(R(X)) \to \mathscr{C}(\mathrm{Spec}_r R(X), \mathbf{Z}/2)$ is an isomorphism.*

1.3. *Unramified cohomology*

Let X be a smooth integral variety over a field k, and let $k(X)$ be its function field. Assume $\mathrm{char}(k) \neq 2$. Let $\mathscr{H}^n$ be the Zariski sheaf associated to the presheaf $U \mapsto H^n(U) = H^n(U_{\acute{e}t}, \mathbf{Z}/2)$. Let $\mathcal{O}_{X,P}$ denote the local ring of the scheme X at the point P and let $\kappa(P)$ be its residue field. The following theorem is a consequence of the results of Bloch and Ogus ([3]):

Theorem 1.3.1. *The following subgroups of $H^n(k(X))$ coincide:*

(a) *The group $H^0(X, \mathscr{H}^n)$ of global sections of the sheaf $\mathscr{H}^n$.*

(b) *The group of elements α in $H^n(k(X))$ such that for each point P of the scheme X, there exists $\alpha_P \in H^n(\mathcal{O}_{X,P}) = H^n_{\acute{e}t}(\mathcal{O}_{X,P}, \mathbf{Z}/2)$ whose image in $H^n(k(X))$ coincides with α.*

(c) *The group of elements α in $H^n(k(X))$ such that for each codimension 1 point P of the scheme X, there exists $\alpha_P \in H^n(\mathcal{O}_{X,P})$ whose image in $H^n(k(X))$ coincides with α.*

(d) *The group of elements α in $H^n(k(X))$ such that for each codimension 1 point P of the scheme X, $\partial_P(\alpha) = 0$, where $\partial_P(\alpha)$ denotes the image of α in $H^{n-1}(\kappa(P))$ under the map ∂_P arising in the Bloch-Ogus complex.*

Moreover, given α and P as in (b) *or* (c), *the element $\alpha_P \in H^n(\mathcal{O}_{X,P})$ is uniquely defined, the map $H^n(\mathcal{O}_{X,P}) \to H^n(k(X))$ being injective.* □

The subgroup $H^0(X, \mathscr{H}^n) \subset H^n(k(X))$ is called the *unramified subgroup* of $H^n(k(X))$ *with respect to* X. We shall sometimes denote it by $H^n_{nr}(k(X), X)$.

As an easy application of Theorem 1.3.1, we have,

Proposition 1.3.2. *Let X be a smooth, proper, integral k-variety, and let $k(X)$ be its function field. The group $H^0(X, \mathscr{H}^n) = H^n_{nr}(k(X), X)$ coincides with the subgroup of $H^n(k(X))$ consisting of elements α such that for each rank one discrete valuation ring A containing k and having $k(X)$ as its field of fractions, the boundary $\partial_A(\alpha)$ vanishes in $H^{n-1}(\kappa(A))$, where $\kappa(A)$ denotes the residue field of A.* □

In particular, the group $H^n_{nr}(k(X), X)$ is a k-birational invariant of smooth projective integral k-varieties: it does not depend on the particular smooth proper model X/k of the function field $k(X)/k$. We may thus denote it simply by $H^n_{nr}(k(X)/k)$ (cf. [8] §1).

For further use, we mention the following lemma (see [8], Proposition 1.3 and [20], §1).

Lemma 1.3.3. *Let A be a discrete valuation ring, let K be its field of fractions and let κ be its residue field. Assume that 2 is invertible in κ. Let a_i $(i = 1, \ldots, n)$ be in K^*. Assume that all a_i are units in A for $i > 1$, let $\alpha_i \in \kappa^*$ be their images in κ^*, and let r be the valuation of a_1. Then*

$$\partial_A(\{a_1, \ldots, a_n\}) = r\{\alpha_2, \ldots, \alpha_n\} \in H^{n-1}(\kappa) \ .$$ □

Proposition 1.2.1 has the following consequence:

Proposition 1.3.4. *Let X/R be a smooth integral variety over a real closed field R, and let $U \subset X$ be a non-empty Zariski open set such that $U(R) = X(R)$. Let d be the dimension of X. Then for $n \geq d+1$, and any $i \geq 0$, the restriction map induces an isomorphism*

$$H^i(X, \mathscr{H}^n) \xrightarrow{\sim} H^i(U, \mathscr{H}^n) \ .$$

Proof. For any point $P \in X \setminus U$, the closure Z_P of P contains no R-point. Let $R(P)$ be the residue field of X at P (which is the function field of Z_P). By Proposition 1.2.1, $H^k(R(P)) = 0$ for any $k > \dim Z_P$. This implies that the global sections of the Bloch–Ogus resolutions of the sheaves $\mathscr{H}^n$ over X and over U coincide as soon as $n \geq d+1$. □

Corollary 1.3.5. *Let $X/\mathbf{R}$ be a smooth geometrically connected variety of dimension d, and assume that $X(\mathbf{R})$ is compact. Then for any $n \geq d+1$, the subgroup $H^0(X, \mathscr{H}^n)$ of $H^n(\mathbf{R}(X))$ coincides with $H^n_{nr}(\mathbf{R}(X)/\mathbf{R})$.*

Proof. By Hironaka's desingularization theorem, one may realize X as a Zariski open set of a smooth proper $\mathbf{R}$-variety X_c. Since $X(\mathbf{R})$ is compact and dense in $X_c(\mathbf{R})$, it coincides with $X_c(\mathbf{R})$. It only remains to apply the previous proposition. (For X quasi-projective, one could also argue (cf. [4], 10.2.5) that $X(\mathbf{R})$ compact implies that any real valuation of $\mathbf{R}(X)$ trivial on $\mathbf{R}$ has its centre on X). □

1.4. *Quadratic forms over fields*

For any field k (char $(k) \neq 2$), let $W(k)$ be its Witt ring and $I^n k \subset W(k)$ $(n \geqq 0)$ be the powers of the fundamental ideal Ik of forms of even rank. The ideal $I^n k$ is spanned, as a group, by the n-dimensional Pfister forms $\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \ldots \otimes \langle 1, -a_n \rangle (a_i \in k^*)$. As explained in [1], Sylvester's theorem yields a signature map:

$$\text{sgn}: W(k) \to \mathscr{C}(\text{Spec}_r\, k, \mathbf{Z})$$

which induces maps

$$\text{sgn}_n: I^n k \to \mathscr{C}(\text{Spec}_r\, k, 2^n\mathbf{Z}) .$$

These maps are compatible for various n with respect to inclusions and thus induce maps

$$I^n k / I^{n+1} k \to \mathscr{C}(\text{Spec}_r\, k, 2^n\mathbf{Z}/2^{n+1}\mathbf{Z}) \simeq \mathscr{C}(\text{Spec}_r\, k, \mathbf{Z}/2\mathbf{Z}) .$$

On the other hand, one checks that the map from $I^n k$ to $I^{n+1} k$ given by multiplication by the form $\langle\langle -1 \rangle\rangle = \langle 1, 1 \rangle$ induces a commutative diagram:

$$\begin{array}{ccc} I^n k / I^{n+1} k & \to & \mathscr{C}(\text{Spec}_r\, k, \mathbf{Z}/2\mathbf{Z}) \\ \downarrow \otimes \langle\langle -1 \rangle\rangle & & \downarrow = \\ I^{n+1} k / I^{n+2} k & \to & \mathscr{C}(\text{Spec}_r\, k, \mathbf{Z}/2\mathbf{Z}) . \end{array}$$

We refer to [1], §2, for the proof of the following theorem.

Theorem 1.4.1. *Let $K = R(X)$ be the function field of a variety X of dimension d over the real closed field R. Let $n \geqq d + 1$.*

(a) *Multiplication by $\langle 1, 1 \rangle$ induces an isomorphism*

$$I^n K \simeq I^{n+1} K .$$

(b) *If K is not formally real, then $I^n K = 0$.*

(c) *The map $I^n K / I^{n+1} K \to \mathscr{C}(\text{Spec}_r\, K, \mathbf{Z}/2\mathbf{Z})$ defined above is an isomorphism.* □

1.5. *Quadratic forms over real function fields and cohomology*

Let A be a rank one discrete valuation ring, let K be its field of fractions and κ its residue field (char $(\kappa) \neq 2$). Associated to a uniformizing parameter π, there is a well-known second-residue homomorphism ∂_π: $W(K) \to W(\kappa)$ which maps $I^{n+1} K$ to $I^n \kappa$ for any n. As is easily checked, the subgroup $\text{Ker}\, \partial_\pi$ does not depend on the choice of the uniformizing parameter π. We shall therefore denote it by $\text{Ker}\, \partial_A$.

If X is a normal integral scheme with 2 invertible in $\mathcal{O}_X$ and with function field $k(X)$, we shall define the *unramified subgroup of $W(k(X))$ with respect to X* as the

intersection

$$W_{nr}(k(X)/X) = \bigcap_{P \in X^{(1)}} \operatorname{Ker} \partial_{A_P} \subset W(k(X)) .$$

Here P runs through all points of codimension 1 of X and A_P denotes the local ring of X at such a point: since X is normal, such a ring is a rank one discrete valuation ring.

For $n \geqq 0$, we define

$$I^n_{nr}(k(X)/X) = I^n k(X) \cap W_{nr}(k(X)/X) \subset W(k(X)) .$$

Proposition 1.5.1. *Let X be a smooth integral R-variety of dimension d and let $R(X)$ be its function field. Let $n \geqq d + 1$.*

(a) *There exists a homomorphism*

$$e_n : I^n R(X) \to H^n(R(X))$$

which is uniquely defined by the requirement that it map the Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ to the cup-product $\{a_1, \ldots, a_n\}$.

(b) *The homomorphism e_n maps $I^n_{nr}(R(X)/X)$ to $H^0(X, \mathscr{H}^n) \subset H^n(R(X))$.*

(c) *The map e_n induces an isomorphism*

$$I^n R(X)/I^{n+1} R(X) \xrightarrow{\sim} H^n(R(X)) ,$$

which is compatible with the isomorphisms of both groups with the group $\mathscr{C}(\mathrm{Spec}_r R(X), \mathbf{Z}/2)$, as recalled in 1.4.1 *and* 1.2.4.

Proof. The proof of (a) and (c) is given in [1], §2. Let P be a point of codimension 1 on X, and let A be the local ring of X at P. Let $\kappa = \kappa(P)$ be the residue field of A, and let π be a uniformizing parameter of A. In order to prove (b), we only have to check that the following diagram is commutative:

$$\begin{array}{ccc} e_n : I^n R(X) & \to & H^n(R(X)) \\ \downarrow \partial_\pi & & \downarrow \partial_A \\ e_{n-1} : I^{n-1}\kappa(P) & \to & H^{n-1}(\kappa(P)) \end{array} .$$

Here the horizontal maps are well-defined according to (a) (the dimension of the function field $\kappa(P)$ is one less than that of $R(X)$), and the vertical maps are the residue maps for Witt groups and the boundary maps in the Bloch–Ogus context. Since $I^n R(X)$ is spanned as an abelian group by the Pfister forms $\langle\langle a_1, \ldots, a_n \rangle\rangle$ with all a_i for $i \geqq 2$ units in the discrete valuation ring A, the commutativity of the above diagram thus follows from the well known formula for the value of the residue homomorphism on such a form and Lemma 1.3.3. □

§2. The main theorem

2.1. Mod 2 *signatures on étale cohomology*

Let A be a commutative ring with $2 \in A^*$. If $(p, <)$ is a point of $\mathrm{Spec}_r A$, there is an associated homomorphism from the residue field $\kappa(p)$ of Spec A at p to the real

closure R^p of the ordered field $\kappa(p)$ and hence a homomorphism from the local ring A_p to R^p. Let $\alpha \in H^n(A) = H^n_{\acute{e}t}(A, \mathbf{Z}/2)$ be an étale cohomology class. There is an associated element $\alpha_p \in \mathbf{Z}/2$, which is the image of α in $H^n(R^p) = \mathbf{Z}/2$. Thus α defines a map:

$$h_n(\alpha): \mathrm{Spec}_r A \to \mathbf{Z}/2$$

$$p \to \alpha_p$$

Lemma 2.1.1. *The map $h_n(\alpha)$ is continuous, i.e. it is locally constant.*

Proof. We may assume $n \geqq 1$. If $\alpha_p \neq 0$, then $\alpha_p = \{-1, \ldots, -1\}_{R^p}$. Changing α into $\alpha - \{-1, \ldots, -1\}_A$, we may assume that $\alpha_p = 0$. If B denotes the strict real henselization of A at the point p of $\mathrm{Spec}_r A$, then $\alpha_B = 0 \in H^n(B)$. By standard limit arguments, we conclude that there exists $f \in A$, with $f(p) > 0$, a ring C, a prime ideal P of C, with an ordered residue field $\kappa(P)$, together with an étale map $A_f \to C$ such that the contraction of P to A_f is p, the extension $\kappa(P)/\kappa(p)$ is finite, the order of $\kappa(P)$ induces the given order of $\kappa(p)$ and such that $\alpha_C = 0 \in H^n(C)$.

The map $\mathrm{Spec}\, C \to \mathrm{Spec}\, A$ being étale, the induced map $\varphi: \mathrm{Spec}_r C \to \mathrm{Spec}_r A$ is an open map (Roy [22], Proposition 2.4). Since the point $p \in \mathrm{Spec}_r A$ belongs to the image of $\mathrm{Spec}_r C$, there exists an open set $U \subset \mathrm{Spec}_r A$ which contains p and such that $U \subset \varphi(\mathrm{Spec}_r C)$. At any point $q \in U \subset \mathrm{Spec}_r A$, the fibre α_q is trivial. □

If $A = K$ is a field, then the map h_n coincides with the map hg_n (cf. §1.1).

Now if $X = \mathrm{Spec}\, A$ is a smooth integral affine variety over R, there is a natural map

$$h_n: H^0(X, \mathscr{H}^n) \to \mathscr{Appl}(\mathrm{Spec}_r A, \mathbf{Z}/2)$$

from $H^0(X, \mathscr{H}^n)$ to the set of maps from $\mathrm{Spec}_r A$ to $\mathbf{Z}/2$, which we now describe. Let $(p, <) \in \mathrm{Spec}_r A$ be a point of the real spectrum of A. Let $A \to R^p$ be the associated homomorphism from A to the real closure R^p of $\kappa(p)$. Any element α of $H^0(X, \mathscr{H}^n) \subset H^n(R(X))$ comes from a well-defined element $\alpha_p \in H^n_{\acute{e}t}(A_p)$. One then defines $h_n(\alpha) \in \mathbf{Z}/2 = H^n(R^p)$ to be the image of α_p under the natural map $H^n_{\acute{e}t}(A_p) \to H^n(R^p)$.

Proposition 2.1.2. *Given any $\alpha \in H^0(X, \mathscr{H}^n)$, $h_n(\alpha)$ is a continuous map,* i.e. *it is a locally constant map.*

Proof. Let $(p, <)$ be a point of $\mathrm{Spec}_r A$. As in the proof of 2.1.1 we may assume that the image of $(p, <)$ in $\mathbf{Z}/2$ is zero. Easy limit arguments show that there exists an open affine neighbourhood $\mathrm{Spec}\, B$ of p in $\mathrm{Spec}\, A$ and an element $\gamma \in H^n(B)$ such that α and γ have the same image in $H^n(R(Y))$. Now given any point q of the smooth R-scheme $\mathrm{Spec}\, B$, the restriction map $H^n(B_q) \to H^n(R(X))$ is injective according to Bloch and Ogus (see 1.3.1 above). Thus on the open set $\mathrm{Spec}_r B \subset \mathrm{Spec}_r A$ the map $h_n(\alpha)$ coincides with the map $h_n(\gamma)$ defined above and the claim follows. □

Thus, if $X = \mathrm{Spec}\, A$ is a smooth integral affine variety over R, we have a map

$$h_n: H^0(X, \mathscr{H}^n) \to \mathscr{C}(\mathrm{Spec}_r A, \mathbf{Z}/2)$$

which we shall call the mod 2 *signature map.* Composition with the inclusion

$X(R) \subset \mathrm{Spec}_r A$ yields a map

$$H^0(X, \mathscr{H}^n) \to \mathscr{C}(X(R), \mathbf{Z}/2) .$$

By the results recalled in 1.1, (resp. by trivial arguments if $R = \mathbf{R}$) the continuity of each $h_n(\alpha)$ on $\mathrm{Spec}_r A$ implies that this map factorizes through $(\mathbf{Z}/2)^S$, where S is the set of semi-algebraic connected components of $X(R)$ (resp. connected components of $X(\mathbf{R})$). We therefore have:

Corollary 2.1.3. *Let X be a smooth integral variety over the real closed field R, and let n be an integer. Let S be the set of semi-algebraic connected components of $X(R)$. Taking fibres at points of $X(R)$ yields a homomorphism*

$$H^0(X, \mathscr{H}^n) \to \mathscr{C}(X(R), \mathbf{Z}/2)$$

which factorises through $(\mathbf{Z}/2)^S$. □

One easily checks that the map h_n is functorial with respect to R-morphisms of smooth integral affine R-varieties. By the remark following Lemma 2.1.1, restriction to the generic point yields a commutative diagram:

$$\begin{array}{ccc} H^0(X, \mathscr{H}^n) & \xrightarrow{h_n} & \mathscr{C}(\mathrm{Spec}_r A, \mathbf{Z}/2) \\ \downarrow & & \downarrow \\ H^n(R(X)) & \xrightarrow{hg_n} & \mathscr{C}(\mathrm{Spec}_r R(X), \mathbf{Z}/2) . \end{array}$$

2.2. *Stability properties for the unramified Witt group*

Proposition 2.2.1. *Let R be a real closed field, and let X be a normal integral R-variety of dimension d. Then for any $n \geqq d+1$, multiplication by $\langle 1, 1 \rangle$ induces an isomorphism*

$$I^n_{nr}(R(X)/X) \simeq I^{n+1}_{nr}(R(X)/X) .$$

Proof. Let P be a point of codimension 1 on X. Its closure Y in X is an integral R-variety of dimension $d-1$. Let π be a uniformizing parameter of the local ring of X at P. There is a commutative diagram:

$$\begin{array}{ccc} I^n R(X) & \to & I^{n+1} R(X) \\ \downarrow & & \downarrow \\ I^{n-1} R(Y) & \to & I^n R(Y) \end{array}$$

whose horizontal maps are given by multiplication by $\langle 1, 1 \rangle$ and vertical maps are given by ∂_π. Since $n \geqq d+1$, Theorem 1.4.1 ensures that the horizontal maps are isomorphisms. Hence multiplication by $\langle 1, 1 \rangle$ induces an isomorphism between the kernels of ∂_π on $I^n R(X)$ and on $I^{n+1} R(X)$. Applying this to all points of codimension 1 on X yields the result. □

Proposition 2.2.2. *Let R be a real closed field and X a smooth integral R-variety of dimension d. Let $U \subset X$ be a non-empty open set. If for each point P of codimension 1 of X not lying in U, the residue field of X at P is not formally real (for instance if $U(R) = X(R)$), and if $n \geqq d+1$, then the natural restriction maps $I^n_{nr}(R(X)/X) \to I^n_{nr}(R(U)/U)$ are isomorphisms.*

Proof. If P is a point of codimension 1 on X not in U, then the residue field $\kappa(P)$ is a non-formally real function field over R of transcendence degree $d-1$. By 1.4.1(b), $I^{n-1}(\kappa(P))=0$, and hence the relevant residue map vanishes on $I^{n-1}(R(X))$. □

If $X=\operatorname{Spec} A$ is a normal, integral, affine scheme with $2\in A^*$, the image of the restriction map $W(A)\to W(k(X))$ lies in $W_{nr}(k(X)/X)$. In view of this remark, Mahé's Theorem (1.1.1) implies the following

Theorem 2.2.3. *Let $X=\operatorname{Spec} A$ be a normal integral affine R-variety (R any real closed field). Let $\operatorname{Spec}_r A=U_1\cup U_2$ be a disjoint union of open sets and $\Omega_i=U_i\cap\operatorname{Spec}_r R(X)$. Then, there exists an element $\alpha\in W_{nr}(R(X)/X)\subset W(R(X))$ and an integer $N\geqq 0$ such that $\operatorname{sgn}(\alpha)$ is 2^N on Ω_1 and 0 on Ω_2.* □

2.3. *Proof of the main theorem*

Theorem 2.3.1. *Let R be a real closed field, and let X be a smooth integral quasi-projective variety over R, of dimension d. Let S be the set of semi-algebraic connected components of $X(R)$. Then for any $n\geqq d+1$, the* mod 2 *signature map yields an isomorphism*

$$H^0(X,\mathscr{H}^n)\simeq(\mathbf{Z}/2)^S.$$

Proof. Since X is quasi-projective and R is not algebraically closed, one may find a non-empty affine open set $U=\operatorname{Spec} A$ of X such that $U(R)=X(R)$ ([5], 1.5). Thus the semi-algebraic connected components of $U(R)$ and of $X(R)$ coincide. On the other hand, the condition $n\geqq d+1$ implies that the inclusion $H^0(X,\mathscr{H}^n)\subset H^0(U,\mathscr{H}^n)$ actually is an isomorphism (Proposition 1.3.4). Since the inclusion $U(R)\subset\operatorname{Spec}_r A$ induces a bijection between the connected components of $\operatorname{Spec}_r A$ and the semi-algebraic components of $U(R)$, we are reduced to proving the following theorem. □

Theorem 2.3.2. *Let R be a real closed field, and let $X=\operatorname{Spec} A$ be an affine, smooth, integral variety over R of dimension d. Let S be the set of connected components of the real spectrum $\operatorname{Spec}_r A$. Then for any $n\geqq d+1$, the* mod 2 *signature map on the real spectrum $\operatorname{Spec}_r A$ yields an isomorphism*:

$$h_n: H^0(X,\mathscr{H}^n)\simeq\mathscr{C}(\operatorname{Spec}_r A,\mathbf{Z}/2)\simeq(\mathbf{Z}/2)^S.$$

Proof. Let $R(X)$ be the field of fractions of A. As in §2.1, there is a commutative diagram

$$\begin{array}{ccc} H^0(X,\mathscr{H}^n) & \to & \mathscr{C}(\operatorname{Spec}_r A,\mathbf{Z}/2) \\ \downarrow & & \downarrow \\ H^n(R(X)) & \to & \mathscr{C}(\operatorname{Spec}_r R(X),\mathbf{Z}/2)\,. \end{array}$$

For $n\geqq d+1$, the bottom map is an isomorphism (Theorem 1.2.4). Hence for such n, the top map also is injective.

Let now $\operatorname{Spec}_r A=U_1\cup U_2$ be a decomposition into two disjoint non-empty open sets. Let $\operatorname{Spec}_r R(X)=\Omega_1\cup\Omega_2$ be the trace of this decomposition on $\operatorname{Spec}_r R(X)$. Since X is smooth, Ω_1 and Ω_2 are non-empty. By Theorem (2.2.3), there exists an integer N and an unramified element $q\in W_{nr}(R(X)/X)$ such that

$\operatorname{sgn}(q) = 2^N$ on Ω_1 and $\operatorname{sgn}(q) = 0$ on Ω_2. Multiplying q by some power of the form $\langle 1, 1\rangle$ if need be, we may assume that q lies in $I^n_{nr}(R(X)/X)$. We then have $N \geqq n$. If $q \in I^{d+1}(R(X))$, then, by Theorem 1.4.1(c), the highest power of the fundamental ideal of $W(R(X))$ in which q lies is detected by its signature. Thus, if $N > n$, Theorem 1.4.1(c) implies that q actually lies in $I^N(R(X))$ and hence in $I^N_{nr}(R(X)/X)$. By stability (Proposition 2.2.1), there exists a form $q_1 \in I^n_{nr}(R(X)/X)$ such that $\langle 1, 1\rangle^{\otimes N-n} q_1 = q$ in $I^N_{nr}(R(X)/X)$. We clearly have $\operatorname{sgn}(q_1) = 2^n$ on Ω_1 and $\operatorname{sgn}(q_1) = 0$ on Ω_2. Let $\alpha = e_n(q_1) \in H^0(X, \mathscr{H}^n)$ be the element given by Proposition 1.5.1(a)(b). Part (c) of that same proposition implies $h_n(\alpha) = 1 \in \mathbf{Z}/2$ on Ω_1 and $h_n(\alpha) = 0 \in \mathbf{Z}/2$ on Ω_2. Since $h_n(\alpha)$ is locally constant on $\operatorname{Spec}_r A$ and $\operatorname{Spec}_r R(X)$ is dense in $\operatorname{Spec}_r A$, X being smooth, we have $h_n(\alpha) = 1 \in \mathbf{Z}/2$ on U_1 and $h_n(\alpha) = 0 \in \mathbf{Z}/2$ on U_2.

Thus the injective map $H^0(X, \mathscr{H}^n) \to \mathscr{C}(\operatorname{Spec}_r A, \mathbf{Z}/2)$ is also surjective, and the theorem is proved. □

2.4. *The compact case*

If $R = \mathbf{R}$ and $X(\mathbf{R})$ is compact, we exhibit an explicit inverse to the signature map. This will be used in §3.

Theorem 2.4.1. *Let X be a smooth integral quasi-projective variety of dimension d over the real field $\mathbf{R}$ such that $X(\mathbf{R})$ is compact. Let S be the set of connected components of $X(\mathbf{R})$. Let $n \geqq d + 1$. Then*

i) *For any $\varepsilon \in \mathscr{C}(X(\mathbf{R}), \mathbf{Z}/2)$, there exists a function $f_\varepsilon \in \mathbf{R}(X)$ which is invertible on $X(\mathbf{R})$ and such that for $x \in X(\mathbf{R})$, $f_\varepsilon(x) > 0$ if and only if $\varepsilon(x) = 0$.*
ii) *The cup-product $\{f_\varepsilon, -1, \ldots, -1\} \in H^n(\mathbf{R}(X))$ does not depend upon the particular choice of f_ε. The map $\varepsilon \mapsto \{f_\varepsilon, -1, \ldots, -1\}$ is the inverse of the* mod 2 *signature map.*

Proof. The same arguments as in the proof of Theorem 2.3.1 allow us to restrict to the case where X is affine, say $X = \operatorname{Spec} A$.

Let $X(\mathbf{R}) = U_1 \cup U_2$ be a decomposition into two disjoint non-empty open sets such that $\varepsilon|_{U_1} = 1$, $\varepsilon|_{U_2} = 0$. According to the Stone–Weierstraß approximation theorem, there exists a function $f \in A$ which is very close to -1 on each point of U_1 and is very close to 1 on each point of U_2. This proves i).

Let P be a point of codimension 1 of the scheme X. The closure $Y \subset X$ of P is an integral $\mathbf{R}$-variety of dimension $(d-1)$. Its function field $\mathbf{R}(Y)$ is the residue field $\kappa(P)$ of X at P. If $\mathbf{R}(Y)$ is formally real, the set $Y(\mathbf{R})$ is not empty (Artin-Lang theorem, see [4], 4.1.2). Since f is invertible at each point of $X(\mathbf{R})$, this implies that f is invertible at P. Since all entries in the cup-product $\alpha = \{f, -1, \ldots, -1\}$ are units, we conclude that $\partial_{A_P}(\alpha) = 0$. On the other hand, if $\mathbf{R}(Y)$ is not formally real, then $H^{n-1}(\mathbf{R}(Y)) = 0$ (Proposition 1.2.1) and hence certainly $\partial_{A_P}(\alpha) = 0$. Thus $\alpha = \{f, -1, \ldots, -1\} \in H^n(\mathbf{R}(X))$ belongs to $H^0(X, \mathscr{H}^n)$. The fibre of α at points of U_1 is $\{-1, -1, \ldots, -1\} = 1 \in \mathbf{Z}/2$ and it is $\{1, -1, \ldots, -1\} = 0 \in \mathbf{Z}/2$ at points of U_2. Thus $h_n(\alpha) = \varepsilon$. The injectivity of h_n, $n \geqq d + 1$ (Theorem 2.3.1) implies assertion ii) of the theorem. □

Remark 2.4.2. Theorem 2.4.1 holds more generally for any archimedean real closed ground field R. On the other hand, for any non-archimedean real closed ground field, Schülting ([23], §5) has examples of surfaces such that functions do not separate semi-algebraic components and Theorem 2.4.1 does not generalise to such ground fields.

Remark 2.4.3. If $X(\mathbf{R})$ is not compact, the theorem breaks down ([5], 1.8.1).

Remark 2.4.4. For X over $\mathbf{R}$ as in Theorem 2.4.1, one wonders whether the map $H^0(X, \mathscr{H}^d) \to H^0(X, \mathscr{H}^{d+1})$ given by cup-product with $\{-1\}$ is surjective. In particular, it is true that the elements of $H^0(X, \mathscr{H}^d)$ separate the connected components of $X(\mathbf{R})$? For curves, this easily follows from the divisibility of the Jacobian of the smooth projective model of $X_{\mathbf{C}}$. For real surfaces satisfying some geometric conditions, this question has a positive answer (Proposition 3.1.2).

§3. Real compact surfaces

3.1. *Degeneration of the Bloch–Ogus spectral sequence*

From the local global spectral sequence

$$E_2^{pq} = H^p(X, \mathscr{H}^q) \Rightarrow H^n(X)\,, \qquad (*)$$

where $H^p(X, \mathscr{H}^q) = H^p(X_{\mathrm{Zar}}, \mathscr{H}^q)$ and the existence of (flasque) Gersten resolutions for the Zariski sheaves $\mathscr{H}^q$, as proved by Bloch/Ogus [3] (which implies that the non-vanishing terms of the spectral sequence $(*)$ are concentrated in the sector $p \leqq q$), one immediately gets the exact sequence:

$$0 \to H^1(X, \mathscr{H}^2) \to H^3(X) \to H^0(X, \mathscr{H}^3) \xrightarrow{d_2} H^2(X, \mathscr{H}^2) \to H^4(X)\,. \qquad (**)$$

Let $CH^2(X)$ denote the Chow group of codimension 2 cycles modulo rational equivalence. The Bloch–Ogus resolution of the sheaf $\mathscr{H}^2$ yields an identification

$$CH^2(X)/2 \simeq H^2(X, \mathscr{H}^2)\,.$$

Presumably, one may check that the map $H^2(X, \mathscr{H}^2) \to H^4(X)$ in the above sequence is induced by the cycle map $CH^2(X) \to H^4(X)$ coming from étale cohomology, but this is not needed in the sequel.

Whenever the differential d_2 in the sequence $(**)$ vanishes, one expects interesting consequences for $CH^2(X)$. However, this differential does not always vanish. The complement of a general quartic in $\mathbb{P}^3_{\mathbf{C}}$ is an example. P. Salberger informs us that d_2 does not vanish for some smooth projective rational surfaces without rational points over certain non-algebraically closed fields. We ask the following questions:

1) Is d_2 zero for smooth projective varieties over algebraically closed fields?
2) Is d_2 zero for smooth projective surfaces over perfect fields provided they have a rational point?

Here is one case where the differential vanishes:

Proposition 3.1.1. *Let $X/\mathbf{R}$ be a smooth geometrically connected surface. Assume that $X(\mathbf{R})$ is compact. Then the Bloch–Ogus spectral sequence $(*)$ degenerates.*

Proof. Since the non-zero terms of the spectral sequence satisfy $p \leqq q$ and $p \leqq 2$, the only differentials in the sequence which might be non-zero are the differentials

$$d_2 \colon H^0(X, \mathscr{H}^i) \to H^2(X, \mathscr{H}^{i-1}) \quad (i \geqq 3) .$$

If $X(\mathbf{R}) = \varnothing$, then $H^i(\mathbf{R}(X)) = 0$ for $i \geqq 3$, hence all differentials vanish. We may therefore assume $X(\mathbf{R}) \neq \varnothing$. From the spectral sequence we get exact sequences ($i \geqq 3$):

$$H^i(X) \to H^0(X, \mathscr{H}^i) \to H^2(X, \mathscr{H}^{i-1}) .$$

Given any Zariski open set $U \subset X$, the functoriality of the spectral sequence gives rise to a commutative diagram of exact sequences

$$\begin{array}{ccccc} H^i(X) & \to & H^0(X, \mathscr{H}^i) & \to & H^2(X, \mathscr{H}^{i-1}) \\ \downarrow & & \downarrow & & \downarrow \\ H^i(U) & \to & H^0(U, \mathscr{H}^i) & \to & H^2(U, \mathscr{H}^{i-1}) . \end{array}$$

Let $U \subset X$ be a Zariski open set such that $U(\mathbf{R}) = X(\mathbf{R})$. Since the cohomological dimension of $\mathbf{C}$ is zero, and the cohomological dimension of the function field of an integral curve C over $\mathbf{R}$ with $C(\mathbf{R}) = \varnothing$ is at most one ([27] or Proposition 1.2.1), it follows that for $i \geqq 3$, the Bloch–Ogus resolutions of the sheaves $\mathscr{H}^i$ over X and over U coincide. Hence for any such U, the two right vertical arrows are isomorphisms for $i \geqq 3$, except possibly for the restriction map $H^2(X, \mathscr{H}^2) \to H^2(U, \mathscr{H}^2)$. This map may be identified with the restriction map $CH^2(X)/2 \to CH^2(U)/2$, hence also with the restriction map $CH_0(X)/2 \to CH_0(U)/2$ (where $CH_0(U)$ denotes the Chow groups of 0-cycles modulo rational equivalence.)

Let Z be a smooth compactification of X. Let F be the closed set $Z \setminus X$ and let S be the set of connected components of $Z(\mathbf{R}) = X(\mathbf{R})$. We have the following commutative diagram:

$$\begin{array}{ccccccc} CH_0(F)/2 & \to & CH_0(Z)/2 & \to & CH_0(X)/2 & \to & 0 \\ & & \downarrow \simeq & & & & \\ & & (\mathbf{Z}/2)^S & & & & \end{array}$$

The top exact sequence is the localisation sequence mod 2. The vertical map is the isomorphism constructed in [7]. By construction, it maps complex closed points to zero. Since $F(\mathbf{R}) = \varnothing$, it follows that $CH_0(Z)/2 \to CH_0(X)/2$ is an isomorphism. The same argument applies to $CH_0(Z)/2 \to CH_0(U)/2$. Thus the restriction map $CH_0(X)/2 \to CH_0(U)/2$ is an isomorphism, hence also the map $H^2(X, \mathscr{H}^2) \to H^2(U, \mathscr{H}^2)$.

Thus, for $i \geqq 3$, if for some open set U as above the map $H^i(U) \to H^0(U, \mathscr{H}^i)$ is surjective, so is the map $H^i(X) \to H^0(X, \mathscr{H}^i)$.

To prove the proposition, it is enough to find such an open set U. As was shown in 2.4, there exists a finite set of functions $f_j \in \mathbf{R}(X)$, $1 \leqq j \leqq N$ which are invertible on $X(\mathbf{R})$ and such that any element of the finite group $H^0(X, \mathscr{H}^i) \subset H^i(\mathbf{R}(X))$, $i \geqq 3$, is a cup-product $\{f_j, -1, \ldots, -1\}$ for some j. Let $U \subset X$ be a Zariski open

set such that $U(\mathbf{R}) = X(\mathbf{R})$ and such that all the functions f_j are invertible on U. The map $H^0(X, \mathscr{H}^i) \to H^0(U, \mathscr{H}^i)$ being an isomorphism, any element of $H^0(U, \mathscr{H}^i)$ is represented by a cup-product $\{f_j, -1, \ldots, -1\}$ for some j, $1 \leqq j \leqq N$, and such a cup-product clearly comes from $H^i(U)$. □

We shall now use the vanishing of d_2 to give a sufficient condition for $H^0(X, \mathscr{H}^2)$ (which is none other than the 2-torsion subgroup of the Brauer group of X) to separate connected components of $X(\mathbf{R})$ (cf. [6], 3.6). This condition is fulfilled by smooth surfaces in $\mathbb{P}^3$ and more generally by complete intersections in projective spaces.

Proposition 3.1.2. *Let $X/\mathbf{R}$ be a smooth surface with $X(\mathbf{R})$ compact. If $H^3_{\acute{e}t}(X_{\mathbf{C}}, \mathbf{Z}/2)$ is zero, then the* mod 2 *signature map*

$$h_2 \colon H^0(X, \mathscr{H}^2) \to (\mathbf{Z}/2)^S$$

is surjective.

Proof. There is an exact sequence of finite étale group schemes over $\mathbf{R}$

$$0 \to (\mathbf{Z}/2)_{\mathbf{R}} \to (\pi_{\mathbf{C}/\mathbf{R}})_*((\mathbf{Z}/2)_{\mathbf{C}}) \xrightarrow{N_{\mathbf{C}/\mathbf{R}}} (\mathbf{Z}/2)_{\mathbf{R}} \to 0 \qquad (***)$$

which induces an exact cohomology sequence

$$H^2(X) \to H^3(X) \to H^3(X_{\mathbf{C}})\,.$$

It is well-known that the connecting homomorphism in this sequence is given by cup-product with $\{-1\}$. Thus, in the following commutative diagram:

$$\begin{array}{ccccc} H^2(X) & \xrightarrow{\cup\{-1\}} & H^3(X) & \to & 0 \\ \downarrow & & \downarrow & & \\ H^0(X, \mathscr{H}^2) & \xrightarrow{\cup\{-1\}} & H^0(X, \mathscr{H}^3) & & \\ & \searrow^{h_2} & \downarrow h_3 \simeq & & \\ & & (\mathbf{Z}/2)^S & & \end{array} \quad ,$$

under the hypothesis of the proposition, the top map is surjective. The map $H^3(X) \to H^0(X, \mathscr{H}^3)$ is surjective by $(**)$ and the previous proposition. This implies the surjectivity of h_2. □

Remark 3.1.3. Let $X/\mathbf{R}$ be a smooth surface with $X(\mathbf{R})$ compact. If X is affine, then $H^3(X_{\mathbf{C}})$ is always zero (cf. [25] p. 51). On the other hand, if $X/\mathbf{R}$ is projective, by duality, $H^3(X_{\mathbf{C}}) = 0$ if and only if ${}_2\mathrm{Pic}\, X_{\mathbf{C}} = 0$.

3.2. *Upper bounds for the number of connected components*

Proposition 3.2.1. *Let $X/\mathbf{R}$ be a smooth geometrically connected surface with $X(\mathbf{R})$ compact. Let S be the set of connected components of $X(\mathbf{R})$ and let s be the cardinality of S. Then:*

(i) *There is a short exact sequence*

$$0 \to H^1(X, \mathscr{H}^2) \to H^3(X) \to H^0(X, \mathscr{H}^3) \to 0\,.$$

(ii) *There is a natural injection* $CH^2(X)/2 \subsetneq H^4(X)$.
(iii) *All the groups* $H^i(X, \mathscr{H}^j)$ *are finite.*
(iv) *The following inequalities hold*:

$$2s \leqq \dim_{\mathbf{Z}/2} H^i(X)\,, \quad i \geqq 4$$
$$2s - 1 \leqq \dim_{\mathbf{Z}/2} H^3(X)\,.$$

Proof. (i) and (ii) follow immediately from the exact sequence (**) and the vanishing of d_2, and (iii) follows from the vanishing of all differentials and the finiteness of the groups $H^i(X)$.

To prove (iv), we may and shall assume $X(\mathbf{R}) \neq \emptyset$.

By using the exact cohomology sequence on function fields deduced from (***) and the vanishing of $H^i(\mathbf{C}(X))$ for $i > \dim X$, we conclude that the cup-product with $\{-1\}$ induces an isomorphism of the Bloch–Ogus resolutions of $\mathscr{H}^i$ with that of $\mathscr{H}^{i+1}$, $i \geqq 3$. Thus $H^2(X, \mathscr{H}^i) \simeq H^2(X, \mathscr{H}^{i+1})$, $i \geqq 3$.

We now show that $H^2(X, \mathscr{H}^2) \xrightarrow{\cup\{-1\}} H^2(X, \mathscr{H}^3)$ is also an isomorphism. Let U be an affine open subset of X such that $U(\mathbf{R}) = X(\mathbf{R})$. Then the map $H^2(X, \mathscr{H}^2) \to H^2(U, \mathscr{H}^2)$ is an isomorphism (cf. Proof of 3.1.1). We have the following commutative diagram:

$$\begin{array}{ccccc} H^2(X, \mathscr{H}^2) & \xrightarrow{\sim} & H^2(U, \mathscr{H}^2) & \subsetneq & H^4(U) \\ \cup\{-1\}\downarrow & & \cup\{-1\}\downarrow & & \cup\{-1\}\downarrow \\ H^2(X, \mathscr{H}^3) & \to & H^2(U, \mathscr{H}^3) & \subsetneq & H^5(U)\,, \end{array}$$

where the right hand side vertical map is injective in view of (***) and the fact that $H^4(U_{\mathbf{C}}) = 0$ ([25], p. 51). Hence the map $H^2(X, \mathscr{H}^2) \xrightarrow{\cup\{-1\}} H^2(X, \mathscr{H}^3)$ is injective and a glance at the resolutions of $\mathscr{H}^2$ and $\mathscr{H}^3$ reveals that it is also surjective. (From these resolutions together with the exact cohomology sequences coming from (***) one can also deduce injectivity, by using the fact that the norm map $CH_0(X_{\mathbf{C}})/2 \to CH_0(X)/2$ is zero – indeed $CH_0(X_{\mathbf{C}})/2$ is spanned by the class of a real point P, and the norm of such a point is $2P$).

We have the following equality for $i \geqq 3$:

$$\dim_{\mathbf{Z}/2} H^2(X, \mathscr{H}^i) = \dim_{\mathbf{Z}/2} H^2(X, \mathscr{H}^2) = \dim_{\mathbf{Z}/2}(CH^2(X)/2) = s\,.$$

Combining this with the fact that $H^0(X, \mathscr{H}^i) \xrightarrow{\sim} (\mathbf{Z}/2)^S$, $i \geqq 3$ (cf. Theorem 2.3.1) and the (non-canonical) decomposition

$$H^i(X) \xrightarrow{\sim} H^0(X, \mathscr{H}^i) \oplus H^1(X, \mathscr{H}^{i-1}) \oplus H^2(X, \mathscr{H}^{i-2})$$

arising from the vanishing of all differentials, the first equality of (iv) follows. In view of the decomposition

$$H^3(X) \xrightarrow{\sim} H^0(X, \mathscr{H}^3) \oplus H^1(X, \mathscr{H}^2)\,,$$

to establish the second inequality, it is enough to prove:

$$\dim_{\mathbf{Z}/2} H^1(X, \mathscr{H}^2) \geqq s - 1.$$

As Bloch [2] has shown, there is an exact sequence:

$$0 \to H^1(X, \mathscr{K}_2)/2 \to H^1(X, \mathscr{K}_2/2) \to {}_2CH^2(X) \to 0 .$$

On the other hand, Merkurev's theorem [18] (see also [19]) implies that there is an isomorphism of Zariski sheaves $\mathscr{K}_2/2 \simeq \mathscr{H}^2$. Thus there is an exact sequence

$$0 \to H^1(X, \mathscr{K}_2)/2 \to H^1(X, \mathscr{H}^2) \to {}_2CH^2(X) \to 0 .$$

(Here and further below ${}_2A$ denotes the 2-torsion subgroup of an abelian group A.)

Now $CH^2(X) = CH_0(X)$. The second inequality will follow if we can prove:

$$s - 1 \leqq \dim_{\mathbf{Z}/2}({}_2CH_0(X)) .$$

We can clearly assume $X(\mathbf{R}) \neq \varnothing$. If X is proper over $\mathbf{R}$, this inequality is proved in [10], §1.3. Assume that X is not proper, and let Z be a smooth $\mathbf{R}$-compactification of X (X is realized as an open set of Z). We have $X(\mathbf{R}) = Z(\mathbf{R})$. Let P be a closed point of Z not in X. The residue field $\kappa(P)$ at P is the complex field $\mathbf{C}$. Let $A_0(Z) \subset CH_0(Z)$ be the subgroup of $CH_0(Z)$ consisting of classes of 0-cycles of degree 0 and let $\tilde{A}_0(X) \subset CH_0(X)$ be the subgroup of $CH_0(X)$ consisting of classes of 0-cycles of even degree (over $\mathbf{R}$). Since any closed point of $Z \setminus X$ has degree 2, this latter subgroup is well defined. The restriction map $A_0(Z) \to \tilde{A}_0(X)$ is surjective.

The surjective map [7] $\varphi: A_0(Z) \to (\mathbf{Z}/2)^{s-1}$ factorizes through the map $A_0(Z) \to \tilde{A}_0(X)$. Indeed, if a 0-cycle on Z is rationally equivalent to 0 on X, then it is rationally equivalent on Z to a 0-cycle which is a linear combination of closed points in $Z \setminus X$, and such points have even degree over $\mathbf{R}$. We thus get a surjective map $\psi: \tilde{A}_0(X) \to (\mathbf{Z}/2)^{s-1}$. The induced map $\mathrm{Ker}(\varphi) \to \mathrm{Ker}(\psi)$ is surjective. Since $\mathrm{Ker}(\varphi)$ is divisible ([7]), it follows that $\mathrm{Ker}(\psi)$ is divisible. Thus the map ${}_2\tilde{A}_0(X) \to (\mathbf{Z}/2)^{s-1}$ induced on 2-torsion is surjective. From the obvious inclusion ${}_2\tilde{A}_0(X) \subset {}_2CH_0(X)$ the claimed inequality now follows. □

Remark 3.2.2. The second inequality in (iv) together with the Hochschild–Serre spectral sequence yields the inequality

$$2s \leqq \sum_{0 \leqq i \leqq 4} \dim_{\mathbf{Z}/2} H^1(G, H^i(X_{\mathbf{C}}))$$

which also follows from Smith theory (cf. Krasnov [16]). Note however that for $X/\mathbf{R}$ proper, better bounds are sometimes given by the inequality

$$4s - 1 \leqq \sum_{0 \leqq i \leqq 4} \dim_{\mathbf{Z}/2} H^i(X_{\mathbf{C}}) + h^{1,1}(X_{\mathbf{C}})$$

obtained by a combination of Smith theory and Hodge theory (Harlamov [13]).

If $H^3(X_{\mathbf{C}}) = 0$, Smith theory ([16]) gives an inequality

$$2s + \dim_{\mathbf{Z}/2} H^1(X(\mathbf{R}), \mathbf{Z}/2) \leqq 2 + \dim_{\mathbf{Z}/2} H^1(G, H^2(X_{\mathbf{C}})) + \dim_{\mathbf{Z}/2} H^1(G, H^1(X_{\mathbf{C}})),$$

whereas the proof of the second inequality in (iv) gives:

$$2s + \dim_{\mathbf{Z}/2} H^1(X, \mathscr{K}_2)/2 \leqq 2 + \dim_{\mathbf{Z}/2} H^1(G, H^2(X_{\mathbf{C}})) + \dim_{\mathbf{Z}/2} H^1(G, H^1(X_{\mathbf{C}})) .$$

Thus any lower bound on $\dim_{\mathbf{Z}/2} H^1(X, \mathscr{K}_2)/2$ leads to a better upper bound for s.

The particular case of rational surfaces will be considered below. In more general cases, Proposition 4.9 of [9] might help.

3.3. *Rational surfaces*

Let $X/\mathbf{R}$ be a smooth projective geometrically connected rational surface, i.e. $X_{\mathbf{C}}$ is $\mathbf{C}$-birational to a projective plane. Then $\operatorname{Pic} X_{\mathbf{C}}$ is a finitely generated $\mathbf{Z}$-free G-lattice ($G = \operatorname{Gal}(\mathbf{C}/\mathbf{R}) = \{1, \sigma\}$). As any such G-lattice, it may be written as a direct sum:

$$\operatorname{Pic} X_{\mathbf{C}} \simeq \mathbf{Z}^a \oplus (\mathbf{Z}[G]/\mathbf{Z}(1+\sigma))^b \oplus \mathbf{Z}[G]^c \quad (a, b, c \in \mathbf{N}),$$

where $\mathbf{Z}[G] = \mathbf{Z} \oplus \mathbf{Z}\sigma$, the integers a, b, c being determined in a unique way by the G-lattice $\operatorname{Pic} X_{\mathbf{C}}$.

Assume $X(\mathbf{R}) \neq \varnothing$. In 1977, Sansuc and the first named author conjectured that $b = 2s - 2$, i.e.,

$$H^1(G, \operatorname{Pic} X_{\mathbf{C}}) \simeq (\mathbf{Z}/2)^{2s-2}.$$

This was later proved by Silhol [26]. As pointed out to us by V.M. Harlamov, a more general result already follows from ([14], Lemma 3.7), together with the Lefschetz fixed point formula. See also Wall ([27], p. 57).

The following proof should therefore just be viewed as an illustration of the methods of the present paper.

By the main theorem, $H^0(X, \mathscr{H}^3) \simeq (\mathbf{Z}/2)^s$. Since $X_{\mathbf{C}}$ is a rational surface, $A_0(X_{\mathbf{C}}) = 0$ and we have ([7]) $A_0(X) \simeq (\mathbf{Z}/2)^{s-1}$, hence ${}_2CH_0(X) \simeq (Z/2)^{s-1}$. On the other hand, there is a natural isomorphism ([6]):

$$H^1(X, \mathscr{K}_2) \simeq \operatorname{Hom}_G(\operatorname{Pic} X_{\mathbf{C}}, \mathbf{C}^*),$$

which induces an isomorphism

$$H^1(X, \mathscr{K}_2)/2 \simeq (\mathbf{Z}/2)^a.$$

Finally, using the isomorphism

$$H^3(X) \xrightarrow{\sim} H^0(X, \mathscr{H}^3) \oplus H^1(X, \mathscr{K}_2)/2 \oplus {}_2CH_0(X),$$

we get

$$H^3(X) \simeq (\mathbf{Z}/2)^{a+2s-1}.$$

Since X is a rational surface, $H^1(X_{\mathbf{C}}) \simeq {}_2\operatorname{Pic} X_{\mathbf{C}} = 0$. By Poincaré duality, we also have $H^3(X_{\mathbf{C}}) = 0$. Since moreover $X(\mathbf{R}) \neq \varnothing$, in the spectral sequence $H^q(G, H^q(X_{\mathbf{C}}, \mathbf{Z}/2)) \Rightarrow H^n(X)$ the map $E_2^{n,0} \to H^n(X)$ is injective. Thus there is a (non-canonical) isomorphism

$$H^3(X) \simeq H^3(G, \mathbf{Z}/2) \oplus H^1(G, H^2(X_{\mathbf{C}})).$$

The rationality of $X_{\mathbf{C}}$ implies that the (2-torsion of the) Brauer group of $X_{\mathbf{C}}$ is trivial. An application of the Kummer sequence gives

$$H^2(X_{\mathbf{C}}) \simeq \operatorname{Pic} X_{\mathbf{C}}/2 \simeq \mathbf{F}_2^{a+b} \oplus \mathbf{F}_2[G]^c,$$

so that

$$H^3(X) \simeq (\mathbf{Z}/2)^{a+b+1} .$$

A comparison of the two values of $H^3(X)$ gives

$$b = 2s - 2 . \quad \square$$

References

1. Arason, J.Kr., Elman, R., Jacob, B.: The graded Witt ring and Galois cohomology. Can. Math. Soc. Conf. Proc. **4**, 17–50 (1984)
2. Bloch, S.: Lectures on algebraic cycles. Duke Univ. Math. Ser. IV (1980)
3. Bloch, S., Ogus, A.: Gersten's conjecture and the homology of schemes. Ann. Sci. Éc. Norm. Supér., 4^e série. **7**, 181–202 (1974)
4. Bochnak, J., Coste, M., Roy, M.-F.: Géométrie algébrique réelle (Ergebnisse der Math. und ihrer Grenzg., 3. Folge, Band 12) Berlin Heidelberg New York: Springer 1987
5. Colliot-Thélène, J.-L.: Formes quadratiques multiplicatives et variétés algébriques. Bull. Soc. Math. France **106**, 113–151 (1978)
6. Colliot-Thélène, J.-L.: Hilbert's theorem for K_2, with application to the Chow groups of rational surfaces. Invent. Math. **71**, 1–20 (1983)
7. Colliot-Thélène, J.-L., Ischebeck, F.: L'équivalence rationnelle sur les cycles de dimension zéro des variétés algébriques réelles. C.R. Acad. Sci. Paris **292**, 723–725 (1981)
8. Colliot-Thélène, J.-L., Ojanguren, M.: Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford. Invent. Math. **97**, 141–158 (1989)
9. Colliot-Thélène, J.-L., Raskind, W.: K_2-cohomology and the second Chow group. Math. Ann. **270**, 165–199 (1985)
10. Colliot-Thélène, J.-L., Sansuc, J.-J., Soulé, C.: Torsion dans le groupe de Chow de codimension deux. Duke Math. J. **50**, 763–801 (1983)
11. Delfs, H., Knebusch, M.: Semi-algebraic topology over a real closed field, I. Math. Z. **177**, 107–129 (1981); II, ibid. **178**, 175–213 (1981)
12. Geyer, W.-D.: Ein algebraischer Beweis des Satzes von Weichold über reelle algebraische Funktionenkörper. In: Algebraische Zahlentheorie, Oberwolfach 1964, Mannheim, Bibliographisches Institut (1967) pp. 83–98
13. Harlamov, V.M.: Real algebraic surfaces, Proceedings of the International Congress of Mathematicians, Helsinski, 1978, I, 421–428
14. Harlamov, V.M.: Topological types of nonsingular surfaces of degree 4 in $\mathbf{RP}^3$. Funct. Anal. Appl. **10**(4), 55–68 (1976)
15. Knebusch, M.: On algebraic curves over real closed field, I. Math. Z. **150**, 49–70 (1976), II, ibid. **151**, 189–205 (1976)
16. Krasnov, V.A.: Harnack–Thom inequalities for mappings of real algebraic varieties. Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983) = Math. USSR Izv. **22**, 247–275 (1984)
17. Mahé, L.: Signatures et composantes connexes. Math. Ann. **260**, 191–210 (1982)
18. Merkur'ev, A.S.: On the norm residue homomorphism of degree 2. Dokl. Akad. Nauk SSSR **261**, 542–547 (1981) = Sov. Math. Dokl. **24**, 546–551 (1981)
19. Merkur'ev, A.S., Suslin, A.A.: K-cohomology of Severi–Brauer varieties and norm residue homomorphism. Izv. Akad. Nauk SSSR, Ser. Mat. **46**, 1011–1046 (1982)
20. Parimala, R.: Witt group of affine three-folds. Duke Math. J. **57**, 947–954 (1988)
21. Pfister, A.: Zur Darstellung definiter Funktionen als Summe von Quadraten. Invent. Math. **4**, 229–237 (1967)
22. Roy, M.-F.: Faisceau structural sur le spectre réel et fonctions de Nash, in Géométrie algébrique réelle et formes quadratiques, pp. 406–432. (Lecture Notes in Mathematics vol. 959). Berlin Heidelberg New York: Springer 1982
23. Schülting, H.-W.: Real holomorphy rings in real algebraic geometry, in Géométrie algébrique réelle et formes quadratiques, pp. 433–442. (Lecture Notes in Mathematics vol. 959). Berlin Heidelberg New York: Springer 1982

24. Serre, J.-P.: Sur la dimension cohomologique des groupes profinis. Topology **3**, 413–420 (1965)
25. SGA 4 1/2: Cohomologie étale; les points de départ, par P. Deligne, rédigé par J.-F. Boutot, (Lecture Notes in Mathematics, vol. 569, pp. 4–75) Berlin Heidelberg New York: Springer 1977
26. Silhol, R.: Cohomologie de Galois et cohomologie des variétés algébriques réelles; applications aux surfaces rationnelles. Bull. Soc. Math. France **115**, 107–125 (1987)
27. Wall, C.T.C.: Real forms of smooth del Pezzo surfaces. J. Reine Angew. Math. **375/376**, 47–66 (1987)
28. Witt, E.: Zerlegung reeller algebraischer Funktionen in Quadrate, Schiefkörper über reellem Funktionenkörper. J. Reine Angew. Math. **171**, 4–11 (1934)

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On Calabi's conjecture for complex surfaces with positive first Chern class

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It is known by classification theory of complex surfaces that $CP^2 \# n\overline{CP^2} (0 \leqq n \leqq 8)$ and $CP^1 \times CP^1$ are only compact differential 4-manifolds on which there is a complex structure with positive first Chern class. In [TY], the authors proved that for any n between 3 and 8, there is a compact complex surface M diffeomorphic to $CP^2 \# n\overline{CP^2}$ such that $C_1(M) > 0$ and M admits a Kähler-Einstein metric. This paper is the continuation of my joint work with professor S.T. Yau [TY]. The main result of this paper is the following.

Main theorem. *Any compact complex surface M with $C_1(M) > 0$ admits a Kähler-Einstein metric if Lie* (Aut(M)) *is reductive.*

This theorem solves one of Calabi's conjectures in case of complex surfaces. The conjecture says that there is a Kähler-Einstein metric on any compact Kähler manifold with positive first Chern class and without holomorphic vector field. Our proof of the above theorem is based on a partial C^0-estimate of the solutions of some complex Monge-Ampére equations we will develop in this paper (Theorem 2.2, Theorem 5.1) and the previous work of the author in [T1] and the joint work with S.T. Yau in [TY].

Let M be a compact Kähler manifold with positive first Chern class and g be a Kähler metric with its associated Kähler class ω_g in $C_1(M)$. Then, the existence of a Kähler-Einstein metric on M is equivalent to the solvability of the following complex Monge-Ampére equations

$$\begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f - t\varphi}\omega_g^n \\ \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \end{cases} \quad \text{on } M \qquad (0.1)_t$$

where $n = \dim_C M$, $\varphi \in C^\infty(M, R^1)$, $0 \leqq t \leqq 1$ and f is a smooth function determined by equations

$$\mathrm{Ric}(g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f \quad \text{and} \quad \int_M e^f \omega_g^n = \int_M \omega_g^n = C_1(M)^n$$

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Since Yau's solution of Calabi's conjecture for Kähler manifolds with vanishing first Chern class, it has been known that the solvability of $(0.1)_t$ for $0 \leqq t \leqq 1$ would follow from an a prior C^0-estimate of the solutions of $(0.1)_t$. The difficulty is that such a C^0-estimate does not exist in general due to those obstructions found by Matsushima [Ma], Futaki [Fu]. In [T1], the author reduces such a C^0-estimate to some integral bound on the solutions of $(0.1)_t$. There are two ways of obtaining such an integral bound on the solutions. One of them is to evaluate the optimal constant of some linearized versions of Moser-Trudinger inequalities for almost plurisubharmonic functions on M as the author did in [T1]. Another is to obtain more informations about the solutions of $(0.1)_t$ and relate the above integral bound of the solutions to the geometry on M, specially, the geometry of plurianticanonical divisors in M. This second approach is our major motivation in this paper to develop an a prior partial C^0-estimate for the solutions of $(0.1)_t$ in case of complex surfaces.

Our partial C^0-estimate for $(0.1)_t$ is based on the following observation. The solvability of $(0.1)_t$ for $0 \leqq t \leqq 1$ does not depend on the choice of a particular Kähler metric g, that is, there is some "gauge" group of the complex Monge-Ampére equations $(0.1)_t$. The author's belief is that such a "gauge" group should play a role in obtaining the C^0-estimate for the solutions of $(0.1)_t$. To understand this "gauge" group, we first recall some natural classes of Kähler metric with its Kähler form in $C_1(M)$. Note that the anticanonical line bundle K_M^{-1} is ample. Therefore, by Kodaira's embedding theorem, for m sufficiently large, the plurianticanonical line bundle K_M^{-1} is very ample, that is, any basis of the group $H^0(M, K_M^{-m})$ gives an embedding of M into some projective space CP^{N_m}, where $N_m + 1 = \dim_C H^0(M, K_M^{-m})$. Then we have a collection C_m of Kähler metrics consisting of the restrictions of the $\frac{1}{m}$-multiple of Fubini-Study metric on CP^{N_m} to the embeddings of M induced by the bases of $H^0(M, K_M^{-m})$. These Kähler metrics are parametrized by the group $PGL(N_m + 1)$. Thus one can consider $PGL(N_m + 1)$ for m large as a "gauge" group of $(0.1)_t$. Now let us see how this group plays a role in our partial C^0-estimate for solutions of $(0.1)_t$. The differences of the metrics in $\mathscr{C}_m$ from the fixed Kähler one g provide a natural set $\mathscr{C}'_m$ of smooth functions ψ in $C^\infty(M, R^1)$ with $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi > 0$ on M and $\sup_M \psi = 0$. One can regard the functions in C'_m as the generalized polynomials on M of degree m. For instance, if $M = CP^n$, then the sections in $H^0(M, K_M^{-m})$ correspond one-to-one to the homogeneous polynomials of degree m and any function in $\mathscr{C}'_m$ is determined by a basis of the linear space of all homogeneous m-polynomials. We now propose the following estimate for the solution of $(0.1)_t$: there is a $m > 0$, depending only on the geometry of M, such that for any solution φ of $(0.1)_t$, there is a function ψ in $\mathscr{C}'_m$ satisfying

$$\|\varphi - \sup_M \varphi - \psi\|_{C^0(M)} \leqq C \tag{0.2}$$

where C is a constant independent of φ, t. In particular, (0.2) implies that for any solution φ of $(0.1)_t$, there is a subvariety $V_\varphi \subset M$ away from which $\varphi - \sup_M \varphi$ is

uniformly bounded and the degree V_φ with respect to K_M^{-1} is bounded independent of φ. Therefore, we can call (0.2) a partial C^0-estimate for $(0.1)_t$.

We further observe (cf. Lemma 2.2) that (0.2) is equivalent to the following

$$\log\left(\sum_{\nu=0}^{N_m} \|S_\nu^m\|_{g_t}^2\right) \geqq C' \tag{0.3}$$

where C' is a constant depending only on the geometry of M, the metric g_t is given by its Kähler form $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$, $\|\cdot\|_{g_t}$ is the hermitian metric of K_M^{-m} with $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ as its curvature form and $\{S_\nu^m\}_{0\leqq\nu\leqq N_m}$ is an orthonormal basis with respect to the inner product induced by g_t and $\|\cdot\|_{g_t}$. To prove (0.3), it suffices to construct plurianticanonical sections on M with its norm bounded from below at any assigned point. In this paper, it is done for $t = 1$ and complex surfaces, i.e., $n = 2$, by using L^2-estimate for $\bar{\partial}$-operators, Gromov's compactness theorem and Uhlenbeck's theory for Yang-Mills connections (cf. Theorem 2.2, 5.1). Moreover, we can take m in either (0.2) or (0.3) to be less than 7 for $t = 1$ and on compact complex surfaces with positive first Chern class. In general, we believe that the estimate (0.3) is also true on higher dimensional Kähler manifolds with positive first Chern class. Note that the group $PGL(N_m + 1)$ contains the automorphism group $\mathrm{Aut}(M)$ of M. The obstructions of Matsushima and Futaki are from this latter group.

Next, we assume that M is always a complex surface with positive first Chern class. In order to prove the existence of Kähler-Einstein metric on M, we need to evaluate the supremum $\alpha_m(M)$ of those exponents α such that the function $\exp(-\alpha\psi)$ for ψ in $\mathscr{C}'_m$ are uniformly L^1-bounded, where $m \leqq 6$ appears in the partial C^0-estimate (0.2). If such an $\alpha_m(M)$ is strictly larger than 2/3, then by [T1] and the partial C^0-estimate (0.2) for solutions of $(0.1)_1$, there is a Kähler-Einstein metric on M. But sometimes the number $\alpha_m(M)$ could be exactly 2/3, then we need to further study the generalized polynomial functions ψ in the partial C^0-estimate (0.2) for the solutions of $(0.1)_1$ and improve the main theorem in [T1] (cf. section 2 for details). All these lead to the proof of our main theorem stated above.

The organization of this paper is as follows. In section 1, we give some preliminary discussions and reduce the proof of our main theorem to some a prior C^0-estimate for the solutions of complex Monge-Ampére equations. The arguments here are standard. In section 2, we prove our main theorem under the assumption of Theorem 2.2 (strong partial C^0-estimate). Some interesting improvements of the main result in [T1] are given. In sections 3 and 4, we begin our first step of the proof of the strong C^0-estimate, i.e., Theorem 2.2. Gromov's compactness theorem and Uhlenbeck's theory for Yang-Mills connections are applied for this purpose. In section 5, we use Hömmander's L^2-estimate for $\bar{\partial}$-operators on plurianticanonical line bundles to prove a weaker version of Theorem 2.2, i.e., partial C^0-estimate stated in either (0.2) or (0.3). We also apply L^2-estimate for $\bar{\partial}$-operators to making reductions of the singular points of some 2-dimensional Kähler-Einstein orbifolds. In particular, we prove that if a Kähler-Einstein orbifold

is the limit of some sequence of Kähler-Einstein surfaces with positive scalar curvature, then it has at most some Hirzebruch-Jung singularities of special type besides rational double points (Theorem 5.2). In sections 6, 7, by studying the plurianticanonical divisors of some rational surfaces, we complete the proof of Theorem 2.2 (strong partial C^0-estimate). There are two appendices, in which we prove one lemma (Lemma 2.4) and one proposition (Proposition 2.1) stated in section 2. The lemma concerns the singularities of plurianticanonical divisors on a complex surface with positive first Chern class and diffeomorphic to either $CP^2 \# 5\overline{CP^2}$ or $CP^2 \# 6\overline{CP^2}$. The proposition should be a classical and elementary result. In the course of the proof of our main theorem, we also obtain some results on the degeneration of Kähler-Einstein surfaces (Theorem 7.1). We refer readers to the end of section 7 for details.

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1. Preliminaries

Let M be a complex surface with positive first Chern class $C_1(M)$. It is known (cf. [GH]) that M is of form either $CP^1 \times CP^1$ or $CP^2 \# n\overline{CP^2}$ $(0 \leqq n \leqq 8)$, i.e., the surface obtained by blowing up CP^2 at n generic points, where "generic" means that no three of these points are colinear and no six of them are on the same quadratic curve. As symmetric spaces, $CP^1 \times CP^1$ and CP^2 admit the standard invariant metrics. These invariant metrics are Kähler-Einstein metrics. An easy computation shows that for $n = 1$ or 2, $CP^2 \# n\overline{CP^2}$ has non-trivial holomorphic vector fields and the Lie algebra of these holomorphic vector fields is not reductive. Thus Matsushima's theorem [Ma] excludes the existence of Kähler-Einstein metrics on these $CP^2 \# n\overline{CP^2}$ $(n = 1, 2)$. The following theorem is proved in [TY] by estimating the lower bound of the holomorphic invariant introduced in [T1].

Theorem 1.1. *For each integer n between* 3 *and* 8, *there is a complex surface M of form* $CP^2 \# nCP^2$ *such that M admits a Kähler-Einstein metric with positive scalar curvature.*

Remark. By a completely different method, Professor Siu [Si] also proved the existence of Kähler-Einstein metrics on $CP^2 \# 3\overline{CP^2}$ and Fermat surface in CP^3. His proof requires that the manifolds considered must have many symmetries, while ours does not.

Denote by $\mathfrak{I}_n$ the collection of all complex surfaces of form $CP^2 \# n\overline{CP^2}$ with positive first Chern class, in other words, those complex structures on the under-

lying differential manifold $CP^2 \# n\overline{CP^2}$ such that the first Chern class is positive. One can easily prove that for $n = 3$ or 4, $\mathfrak{I}_n$ consists of only one element, i.e. the one on which there is a Kähler-Einstein metric constructed in Theorem 1.1. Hence, in order to prove our main theorem, we may assume that $5 \leqq n \leqq 8$.

Lemma 1.1. *$\mathfrak{I}_n$ is connected in the sense that for any M, M' in $\mathfrak{I}_n$, there is a family of $\{M_t\}_{0 \leqq t \leqq 1}$ such that $M_0 = M$, $M_1 = M'$, $M_t \in \mathfrak{I}_n$ and M_t depends smoothly on t.*

Proof. By induction, we may assume that M, M' are the surfaces obtained by blowing up CP^2 at generic points $p_1, \ldots, p_n$; $p'_1, \ldots, p'_n$, respectively and $p_i = p'_i$ for $i = 1, \ldots, n-1$. Let D be the union of lines $\overline{p_i p_j}$ in CP^2 for $1 \leqq i, j \leqq n-1$ and the quadratic curves in CP^2 passing through five of $p_1, \ldots, p_{n-1}$. $CP^2 \backslash D$ is connected. Now it is easy to see how to connect M to M' smoothly in $\mathfrak{I}_n$.

Lemma 1.2. *For $n \geqq 5$, each M in $\mathfrak{I}_n$ has no non-trivial holomorphic vector field.*

Proof. It suffices to prove that the identity component $\mathrm{Aut}_0(M)$ of the automorphism group is discrete. It is clear that

$$\mathrm{Aut}_0(M) = \{\sigma \in \mathrm{Aut}(CP^2) | \sigma \text{ interchanges } p_1, \ldots, p_n\}$$

where $p_1, \ldots, p_n$ are the blowing-up points of M in CP^2. Then the lemma follows from a straightforward calculation.

We fix a n between 5 and 8 and let M_0 be the complex surface with Kähler-Einstein metric g_0 in Theorem 1.1. Then M_0 is in $\mathfrak{I}_n$. In order to prove our main theorem, we pick an arbitrary smooth family $\{M_t\}_{0 \leqq t \leqq 1}$ from $\mathfrak{I}_n$. It suffices to prove that any M_t admits a Kähler-Einstein metric with positive scalar curvature. We will use the continuous method. Let

$$I = \{t \in [0, t] | M'_t \text{ admits Kähler-Einstein metric for } t' \leqq t\} .$$

Then I contains 0, in particular, I is nonempty. We need to prove that I is both open and closed. In this section, we will prove that I is open. The next several sections are devoted to the proof of the closedness. As in [Y1], [T1], etc., we first convert the existence of Kähler-Einstein metrics into solvability of some complex Monge-Amperé equation. Since the entire family $\{M_t\}$ is in $\mathfrak{I}_n$, there is a smooth family of Kähler metrics $\tilde{g}_t$ on M_t with $0 \leqq t \leqq 1$. Let $\omega_{\tilde{g}_t}$ be the associated Kähler form of $\tilde{g}_t$, then in local coordinates (z_1, z_2) of M_t,

$$\omega_{\tilde{g}_t} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{2} \tilde{g}_{tij} dz_i \wedge \overline{dz_j} \quad \text{where} \quad \tilde{g}_t = \sum_{i,j=1}^{2} \tilde{g}_{tij} dz_i \otimes \overline{dz_j}$$

We choose $\tilde{g}_t$ such that its Kähler form $\omega_{\tilde{g}_t}$ represents the first Chern class of M_t. Then by solving a family of elliptic equations, we can have a smooth family of functions $\{f_t\}_{0 \leqq t \leqq 1}$ such that

$$\mathrm{Ric}(\tilde{g}_t) = \omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f_t, \quad \int_{M_t} e^{f_t} dV_{\tilde{g}_t} = 9 - n$$

where $dV_{\tilde{g}_t} = \omega_{\tilde{g}_t}^2 = \omega_{\tilde{g}_t} \wedge \omega_{\tilde{g}_t}$ is the associated volume form of $\tilde{g}_t$ and $\mathrm{Ric}(\tilde{g}_t)$ is the Ricci form, i.e., in local coordinates (z_1, z_2), if we denote by (R_{tij}) the Ricci curvature

tensor, then

$$\mathrm{Ric}(\tilde{g}_t) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{2} R_{ij\bar{l}} dz_i^k \wedge \overline{dz_j}$$

It is well-known that the existence of Kähler-Einstein metrics on M_t is equivalent to the solvability of the following complex Monge-Amperé equation on M_t (cf. [Y1]).

$$\begin{cases} \left(\omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^2 = e^{f_t - \varphi}\omega_{\tilde{g}_t}^2 \\ \left(\omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \end{cases} \tag{1.1}_t$$

We remark that if $(1.1)_t$ has a solution φ_t, then the corresponding Kähler-Einstein metric has $\omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$ as its Kähler form.

We first prove the openness of I by applying Implicit Function Theorem to the equation $(1.1)_t$.

Lemma 1.3. *Let $\{M_t\}$, I be defined as above. Then I is open.*

Proof. Let $(1.1)_{t_0}$ has a solution φ_{t_0}. The linearized operator of $(1.1)_t$ at $t = t_0$ is $L_{t_0} = \Delta_{t_0} - id$, i.e., for any smooth function v, $L_{t_0}v = \Delta_{t_0}v - v$, where Δ_{t_0} is the Laplacian associated to the Kähler-Einstein metric g_{t_0} on M_{t_0}.

By Lemma 1.2 and the standard Bochner's formula, one can prove that the first nonzero eigenvalue of Δ_{t_0} is strictly greater than one (cf. [Au]). It follows that the linearized operator L_{t_0} is invertible. Then this lemma follows from Implicit Function Theorem.

Therefore, it suffices to prove that I is closed in the interval $[0, 1]$. Without losing the generality, we may assume that $[0, 1) \in I$. By the standard elliptic theory, to show that $1 \in I$, it suffices to prove the uniform C^3-estimate of the solutions of the equations $(1.1)_t$.

Lemma 1.4. *There is a constant C independent of t such that for any solution φ of some equation $(1.1)_t$, we have*

$$\sup_{x \in M_t} \{\|\nabla_t \varphi\|_{\tilde{g}_t}(x), \|\nabla_t^2 \varphi\|_{\tilde{g}_t}(x), \|\nabla_t^3 \varphi\|_{\tilde{g}_t}(x)\} \leqq C \sup_{x \in M_t} \{|\varphi(x)|\}$$

where ∇_t is the gradient with respect to the metric $\tilde{g}_t$ and $|\cdot|_{\tilde{g}_t}$ is the norm induced by $\tilde{g}_t$.

Proof. It follows from same computations as the corresponding ones in [Y1].

By this lemma, we only need to prove the uniform C^0-estimate of the solutions of equations $(1.1)_t$. This will be done in the following sections. We should mention that obtaining such an a prior C^0-estimate is the hardest part of solving this conjecture of E. Calabi.

2. The proof of main theorem

In this section, we will prove our main theorem under the assumption of Theorem 2.2. We postpone the proof of this theorem to the next several sections due to its length. Basically, this theorem provides us a strong partial C^0-estimate of the solutions of the complex Monge-Amperé equations $(1.1)_t$. Let $\{M_t\}_{0 \leqq t \leqq 1}$ be the smooth family in $\mathfrak{I}_n$ given in last section, where $5 \leqq n \leqq 8$. Then each M_t for $t < 1$ has a Kähler-Einstein metric g_t with $\mathrm{Ric}(g_t) = \omega_{g_t}$. It follows that each equation $(1.1)_t$ for $t < 1$ has a solution φ_t. Note that such a solution φ_t is unique by Bando and Mabuchi's uniqueness theorem [BM]. By the discussions in last section, in order to prove the main theorem, it suffices to show the uniform C^0-estimate of those solutions φ_t.

In [T1], the author reduces the C^0-estimate of the solution φ_t to the evaluation of some integral of φ_t. By evaluating this integral, the author proves in [T1] the existence of Kähler-Einstein metrics on Fermat hypersurfaces in CP^{n+1} of degree n or $n+1$ and the authors in [TY] prove Theorem 1.1. Here we develop an effective method to evaluate the integral posed in [T1] for the C^0-estimate of φ_t. We start with the following theorem, which is essentially the main theorem proved in [T1].

Theorem 2.1. *Let $\{M_t\}$ be the family in $\mathfrak{I}_n$ given as above. Then the Kähler manifold $M = M_1$ admits a Kähler-Einstein metric with positive scalar curvature if and only if one of the following holds.*

(1) *There are constants $\varepsilon, C > 0$ and a subsequence $\{t_i\}_{i \geqq 1}$ in the interval $[0, 1)$ with $\lim_{i \to \infty} t_i = 1$, such that for all i and any solution φ_{t_i} of $(1.1)_{t_i}$,*

$$\int_{M_{t_i}} e^{-(2/3+\varepsilon)(\varphi_{t_i} - \sup_{M_{t_i}} \varphi_{t_i})} dV_{\tilde{g}_{t_i}} \leqq C \tag{2.1}$$

(2) *There are constants $\varepsilon, C > 0$ and a subsequence $\{t_i\}_{i \geqq 1}$ in the interval $[0, 1)$ with $\lim_{i \to \infty} t_i = 1$, such that for all i and any solution φ_{t_i} of $(1.1)_{t_i}$,*

$$-\inf_{M_{t_i}} \varphi_{t_i} \leqq (2-\varepsilon) \sup_{M_{t_i}} \varphi_{t_i} + C \tag{2.2}$$

and moreover for any $\lambda < \frac{2}{3}$, there is a constant $C(\lambda)$, depending only on λ, such that

$$\int_{M_{t_i}} e^{-\lambda(\varphi_{t_i} - \sup_{M_{t_i}} \varphi_{t_i})} dV_{\tilde{g}_{t_i}} \leqq C(\lambda) \tag{2.3}$$

Proof. Consider complex Monge-Amperé equations

$$\begin{cases} \left(\omega_{\tilde{g}_t} + \dfrac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi\right)^2 = e^{f_t - s\varphi} \omega_{\tilde{g}_t}^2 \\ \left(\omega_{\tilde{g}_t} + \dfrac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi\right) > 0 \end{cases} \tag{2.4}_{t,s}$$

where $s \leqq 1$. By Lemma 1.2 and uniqueness theorem in [BM], the solution of

equation $(1.1)_t$ for each t is unique. Thus by the same argument as in [BM] for the proof of the uniqueness, one can produce a family $\{\varphi_{t,s}\}_{0 \leq t,s<1}$ of smooth functions, such that the function $\varphi_{t,s}$ solves the equation $(2.4)_{t,s}$ and $\lim_{s \to 1} \varphi_{t,s} = \varphi_t$. Thus in case (1), the theorem follows from the proof of the main theorem in [T1].

In the case (2), choose λ between $\dfrac{2-\varepsilon}{3-\varepsilon}$ and $\dfrac{2}{3}$, then as in [T1], by the concavity of the logarithm function, we have

$$\sup_{M_{t_i}} \varphi_{t_i} \leqq \frac{1-\lambda}{\lambda} \int_{M_{t_i}} (-\varphi_{t_i}) dV_{g_{t_i}} + C'(\lambda) \tag{2.5}$$

where $C'(\lambda)$ is a constant depending only on λ. Combining this with (2.2), we obtain the uniform C^0-estimate of the solution φ_{t_i}. The theorem follows. We refer readers to [T1] for more details.

By this theorem, we see that, to prove the main theorem, it suffices to find a subsequence $\{t_i\}$ having the estimates either (2.1) or (2.2), (2.3).

For $t < 1$, we choose an orthonormal basis $\{S^t_{m\beta}\}_{0 \leq \beta \leq N_m}$ of the group $H^0(M_t, K^m_{M_t})$ with respect to the metric g_t, when $m \geqq 1$ and $N_m + 1$ is the dimension of $H^0(M_t, K^m_{M_t})$. By Kodaira's embedding theorem, those bases $\{S^t_{m\beta}\}_{0 \leq \beta \leq N_m}$ define embeddings $\Phi(t, m)$ from M_t into CP^{N_m} for m large. In fact, when $n = 5, 6$, the embeddings $\Phi(t, 1)$ are well-defined.

Theorem 2.2. (Strong partial C^0-estimate). *There are constants $c(n, m) > 0$, depending only on n, m, and a subsequence $\{t_i\}$ in the interval $[0, 1)$ with $\lim_{i \to \infty} t_i = 1$, such that for $m = 6k$ $(k \geqq 1)$ in case $n = 5, 6$, and $m = 2k$ $(k \geqq 1)$ in case $n = 7, 8$,*

$$\inf_{x \in M_{t_i}} \left\{ \sum_{\beta=0}^{N_m} \|S^{t_i}_{m\beta}\|^2_{g_{t_i}}(x) \right\} \geqq c(n, m) \tag{2.6}$$

where $\|\cdot\|_{g_i}$ is the norm on the line bundle $K^{-m}_{M_t}$ induced by the metric g_t.

Remark. In case $n = 5, 6, 7$, the estimate (2.6) should also hold for $m = 1$. This can be used to simplify the proof of our main theorem, but the simplification is not substantial.

The proof of Theorem 2.2 will be given in the following sections. We will first prove a weak version of this theorem, i.e., Theorem 5.1, in §5. Then we deduce Theorem 2.2 from that weak version.

Let us now see the implications of Theorem 2.2. For each $t \in [0, 1)$, we further choose an orthonormal basis $\{\tilde{S}^t_{m\beta}\}_{0 \leq \beta \leq N_m}$ of $H^0(M_t, K^{-m}_{M_t})$ with respect to the metric $\tilde{g}_t$. Such a basis gives an embedding $\Psi(t, m)$ of M_t into CP^{N_m} whenever the basis $\{S^t_{m\beta}\}$ does. Two embeddings $\Psi(t, m)$, $\Phi(t, m)$ are different by an automorphism $\sigma(t, m)$ in CP^{N_m}, i.e. $\sigma(t, m) \in PGL(N_{m+1})$ and

$$\Phi(t, m) = \sigma(t, m) \circ \Psi(t, m) \tag{2.7}$$

By changing the orthonormal bases $\{\tilde{S}^t_{m\beta}\}_{0 \leq \beta \leq N_m}$, $\{S^t_{m\beta}\}_{0 \leq \beta \leq N_m}$ if necessary, we may assume that each $\sigma(t, m)$ is represented by a diagonal matrix $\operatorname{diag}(\lambda_j(t))_{0 \leq j \leq N_m}$ with $0 < \lambda_0(t) \leqq \ldots \leqq \lambda_{N_m}(t) = 1$.

The following lemma is actually an observation on which the whole proof is based.

Lemma 2.1. *Let M_t, φ_t, $\{S^t_{m\beta}\}_{0\leq\beta\leq N_m}$, $\{\tilde{S}^t_{m\beta}\}_{0\leq\beta\leq N_m}$ be given as above, where $t<1$. Then for $m=6k(k\geqq 1)$ in case $n=5,6$ or $m=2k$ $(k\geqq 1)$ in case $n=7,8$,*

$$\varphi_t = f_t - \frac{1}{m}\log\left(\sum_{\beta=0}^{N_m}\|S^t_{m\beta}\|^2_{\tilde{g}_t}\right)+\frac{1}{m}\log\left(\sum_{\beta=0}^{N_m}|\lambda_\beta(t)|^2\|\tilde{S}^t_{m\beta}\|^2_{\tilde{g}_t}\right)+a(t) \tag{2.8}$$

where $a(t)$ is a constant depending only on t.

Proof. We remark that under the assumption on m, the group $H^0(M_t, K^{-m}_{M_t})$ is free of base point. Thus the left-handed side of (2.8) is a well-defined function on M_t. We denote it by φ'_t. One can check that φ'_t satisfies the equation

$$\omega_{g_t}=\omega_{\tilde{g}_t}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi'_t$$

On the other hand, the solution φ_t of $(1.1)_t$ also satisfies the same equation. The lemma follows.

From now on, in this section, we fix the subsequence $\{t_i\}$ in Theorem 2.2. For simplicity, in the following, we will replace t_i by i whenever t_i appear as subscripts or superscripts. The following lemma is a corollary of Theorem 2.2 and also explains why we call (2.6) in Theorem 2.2 "partial C^0-estimate".

Lemma 2.2. *Let $\{(M_i, g_i)\}$ be the subsequence of Kähler-Einstein manifolds given in Theorem 2.2. Define $\nu=6$ in case $n=5,6$ and $\nu=2$ in case $n=7,8$. Then there is a constant C independent of i, such that for any solution φ_i of $(1.1)_{t_i}$,*

$$\sup_{M_i}\left|\varphi_i-\sup_{M_i}\varphi_i-\frac{1}{\nu}\log\left(\sum_{\beta=0}^{N_\nu}|\lambda_\beta(i)|^2\|\tilde{S}^i_{m\beta}\|^2_{\tilde{g}_t}\right)\right|\leqq C \tag{2.9}$$

where $\{\lambda_\beta(i)\}_{0\leq\beta\leq N_\nu}$ are defined by those automorphisms $\sigma(t_i,\nu)$ in (2.7).

Proof. Note that both metrics $\tilde{g}_i$ (resp. functions f_i) converge to a smooth metric $\tilde{g}$ (resp. a smooth function f) equal to $\tilde{g}_t$ (resp. f_t) with $t=1$. Then the lemma follows from Lemma 2.1 and Theorem 2.2.

Remark. This lemma implies that the normalizations $\varphi_i-\sup_{M_i}\varphi_i$ are uniformly bounded away from some subvarieties in M_i. One should be able to derive from it that $\varphi_i-\sup_{M_i}\varphi_i$ either are uniformly bounded or converge to a function G on $M\backslash D$, where $M=\lim M_i$ and D is a subvariety of M contained in anticanonical divisors, such that G satisfies the equation $\left(\omega_{\tilde{g}}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}G\right)^2=0$ outside D and has logarithmic singularity along D. This singular function G can be regarded as a Green function of the complex Monge-Amperé operator. This Green function must impose some analytic structures on M. Hopefully, by studying these structures, one can determine when φ_i converge to a bounded function on M.

Let ν be defined as in the above lemma. We define rational integrals $I(\alpha, i)$ as follows,

$$I(\alpha,i)=\int_{M_i}\left(\sum_{\beta=0}^{N_\nu}|\lambda_\beta(i)|^2\|\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}(z)\right)^{-\alpha/\nu}dV_{\tilde{g}_i}(z) \tag{2.10}$$

Then by Lemma 2.2, we see that the estimates (2.1) and (2.3) are equivalent to $I(\alpha, i) \leqq C$ for $\alpha = 2/3 + \varepsilon$ and λ, respectively. Thus we are bound to estimate the integrals in (2.10). This will be done in the following lemmas.

Let $\{\tilde{S}_\beta\}_{0 \leqq \beta \leqq N_\nu}$ be the limit of the bases $\{\tilde{S}^i_{\nu\beta}\}_{0 \leqq \beta \leqq N_\nu}$ as i goes to infinity. Then this limit is an orthonormal basis of $H^0(M, K_M^{-\nu})$ with respect to the metric $\tilde{g}$, where $M = M_t$ for $t = 1$. By taking the subsequence if necessary, we may assume that $\lim_{i \to \infty} \lambda_\beta(i) = \lambda_\beta \geqq 0$ for each β. Note that $\lambda_{N_\nu} = 1$ and $\lambda_0 \leqq \ldots \leqq \lambda_{N_\nu}$.

Lemma 2.3. *We adopt the notations given above. Let $N = N_\nu$. Then we have the following estimates,*

(i) *if $n = 8$, then $I(\alpha, i) \leqq C_\alpha$ for any $\alpha < \frac{5}{6}$.*
(ii) *if $n = 7$, then $I(\alpha, i) \leqslant C_\alpha$ for any $\alpha < \frac{3}{4}$.*

Proof. Since the proof for (ii) is identical to that for (i), we only prove (i) here. Put

$$\psi_i = \frac{1}{\nu} \log \left(\sum_{\beta=0}^{N} |\lambda_\beta(i)|^2 \, \| \tilde{S}^i_{\nu\beta} \|^2_{\tilde{g}_i} \right)$$

Denote by $\tilde{\omega}_i$ the Kähler form associated to the metric $\tilde{g}_i$. Then $\tilde{\omega}_i + \partial\bar{\partial}\psi_i$ define positive, d-closed, (1.1)-currents ω'_i on M_i. When i tends to infinity, we may assume that ω'_i converge weakly to a positive, d-closed, (1.1)-current $\frac{\sqrt{-1}}{2\nu\pi}\partial\bar{\partial} \log(|\tilde{S}_N|^2)$ denoted by ω'_∞, where $|\cdot|$ is the absolute value. Now $I(\alpha, i)$ is just the integral $\int_{M_i} e^{-\alpha\psi_i} dV_{\tilde{g}_i}$. By the discussion in §2 of [TY], we have

Fact (†). If $z_i \in M_i$ with $\lim_{i \to \infty} z_i = z \in M$ and the Lelong number $L_{\tilde{g}}(\omega'_\infty, z) \leqq 1$, then there is a $r > 0$ independent of i, such that

$$\int_{B_r(z_i, \tilde{g}_i)} e^{-\alpha\psi_i} dV_{\tilde{g}_i} \leqq C_\alpha \quad \text{for any} \quad \alpha < 1 \tag{2.11}$$

where $B_r(z_i, \tilde{g}_i)$ is the geodesic ball in M_i with radius r and the center at z_i. For more about the Lelong number, one can refer to [Le].

Let D be the zero divisor of $\tilde{S}_N$. If z is either in the complement of D or a point of D with multiplicity 2, the Lelong number $L_{\tilde{g}}(\tilde{\omega}_\infty, z) \leqq 1$. Therefore, we only need to estimate the integral in (2.11) near those points $z_i \in M_i$ with $\lim_{i \to \infty} z_i = z$ being a singular point of D with multiplicity $\geqq 3$.

Since $C_1(M)^2 = 1$, D has no singular point with the multiplicity greater than 2 if it is irreducible. It implies that the Lelong number $L_{\tilde{g}}(\tilde{\omega}_\infty, z) \leqq 1$ at every point z in M if D is irreducible. So we may write $\tilde{S}_N = S'_1 \cdot S'_2$. Since $C_1(M)^2 = 1$, one can easily derive (2.11) by Fact (†) so long as S'_1 is not colinear to S'_2. So we may assume that $S'_1 = S'_2$ and both S'_1, S'_2 are in $H^0(M, K_M^{-1})$. Also by Fact (†), it suffices to estimate the integral in (2.11) at the singular points of the divisor $\{x \in M | S'_1(x) = 0\}$. Let $\pi_i : M_i \mapsto CP^2$, $\pi : M \mapsto CP^2$ be the natural projections induced by blowing-ups. Then each $\pi_{i*}(\tilde{S}^i_N)$ is a sextic curve in CP^2, converging to $\pi_*(\tilde{S}_N) = \pi_*(S'_1)^2$ as i goes to infinity. Now $\pi_*(S'_1)$ is an irreducible singular cubic curve in CP^2. Each $\pi_*(S'_i)$ has only one singular point x, which is either an ordinary double point or a cusp, and is not one of blowing-up points. Without losing generality, we may assume that x is a cusp. Let U be a small neighborhood of x in

CP^2, determined later. Then

$$\int_{M_i} e^{-\alpha\psi_i} dV_{\tilde{g}_i} \leqq C\left(1 + \int_U \frac{1}{|f_i|^{2\alpha}} dV\right) \tag{2.12}$$

where f_i is the local holomorphic function defining $\pi_{i*}(\tilde{S}_N^i)$ in U, dV is the volume form of the euclidean metric on U and C is a constant depending only on U. We will always use C to denote a constant independent of i in this proof, although the quantity of this could be changed in different places. We normalize those f_i such that the limit $f = \lim_{i\to\infty} f_i$ exists and defines $\pi_*(\tilde{S}_N)$ in U. Choose local coordinates (z, w) with $x = (0, 0)$ and

$$f = (z^2 - w^3)^2 \tag{2.13}$$

where f is the local defining function of the divisor D in U. By holomorphic transformations of form $(z, w) \mapsto (z + b_1 + b_2 w + b_3 w^2, w)$, we may assume that

$$\begin{aligned} f_i &= f + \sum_{\substack{3k+2l<12 \\ k\neq 3}} a_{kl}(i) z^k w^l + \sum_{3k+2l\geqq 12} a_{kl}(i) z^k w^l \\ &= f + f_{iL} + f_{iR} \end{aligned} \tag{2.14}$$

where $\lim_{i\to\infty} f_{iL} = 0$, $\lim_{i\to\infty} f_{iR} = 0$. Define

$$\delta_i = 40 \max_{3k+2l<12} \{|a_{kl}(i)|^{\frac{1}{12-3k-2l}}\} \tag{2.15}$$

We denote by J_i the integral on the right side of (2.12) and split J_i into three parts J_{i1}, J_{i2}, J_{i3} as follows,

$$\begin{aligned} J_{i1} &= \int_{\substack{|z|^4 \leqq |w|^6 \\ 1 \geqq |w| \geqq \delta_i^2}} \frac{dV}{|f_i|^{2\alpha}} \\ &= \int_{\substack{|w| \leqq 1 \\ |\xi| \leqq 1}} \frac{|w|^8 dV}{|f_i(w^3\xi, w^2)|^{2\alpha}} \\ &= \int_{\substack{|w| \leqq 1 \\ |\xi| \leqq 1}} \frac{|w|^{8-12\alpha} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{|(\xi^2 - 1)^2 + w^{-6}(f_{iL}(w^3\xi, w^2) + f_{iR}(w^3\xi, w^2)|^{2\alpha}} \end{aligned} \tag{2.16}$$

By the definition of f_{iL}, f_{iR} and δ_i, for i sufficiently large, one can easily see

$$|w^{-6}(f_{iL}(w^3\xi, w^2) + f_{iR}(w^3\xi, w^2))| \leqq \tfrac{1}{4} \quad \text{for} \quad \delta_i \leqq |w| \leqq 1, \quad |\xi| \leqq 1 \tag{2.17}$$

It follows that for any fixed w with $|w| \leqq \delta_i$, the holomorphic functions $(\xi^2 - 1) + w^{-6}(f_{iL} + f_{iR})$ have exactly four distinct zeroes in $\{|\xi| \leqq 1\}$ for all i, moreover, these four zeroes are disjoint from each other by a uniform distance. Therefore, we conclude that the integral J_{i1} is uniformly bounded independent of i for $\alpha < \frac{5}{6}$, i.e., $J_{i1} \leqq C_\alpha$. Similarly, we also have

$$J_{i2} = \int_{\substack{|z|^4 \geqq |w|^6 \\ 1 \geqq |z| \geqq \delta_i^3}} \frac{dV}{|f_i|^{2\alpha}} \leqq C_\alpha \quad \text{for any } \alpha < \frac{5}{6} \tag{2.18}$$

It remains to estimate J_{i3}, which is equal to $J_i - J_{i1} - J_{i2}$. By scaling $(z, w) \to (\delta_i^3 z, \delta_i^2 w)$, we have

$$J_{i3} = \int_{\substack{|z| \leqq 1 \\ |w| \leqq 1}} \frac{\delta_i^{10-12\alpha} dV}{|\delta_i^{-12} f_i(\delta_i^3 z, \delta_i^2 w)|^{2\alpha}}$$

Put $g_i = \delta_i^{-12} f_i(\delta_i^3 z, \delta_i^2 w)$, then by taking a subsequence, we may assume that g_i converge to a polynomial g. Note that by the expansion of f_i in (2.14), we have

$$g = (z^2 - w^3)^2 + \sum_{\substack{3k+2l<12 \\ k<3}} b_{kl} z^k w^l \tag{2.19}$$

where b_{kl} are constants and at least one of them is nonzero. This new polynomial g is less singular than the function f. For example, one can compute

$$\int_{\substack{|z| \leqq 1 \\ |w| \leqq 1}} \frac{dV}{|g|^{2\alpha}} < +\infty \quad \text{for any } \alpha < \frac{11}{24} \tag{2.20}$$

while

$$\int_{\substack{|z| \leqq 1 \\ |w| \leqq 1}} \frac{dV}{|f|^{2\alpha}} = +\infty \quad \text{for any } \alpha \geqq \frac{10}{24} = \frac{5}{12} \tag{2.21}$$

Also, the multiplicity of g at any point in $\{|z| < 1, |w| < 1\}$ is less than that of f at the origin. Therefore, by induction we can prove that $J_{i3} \leqq C_\alpha$ for $\alpha < \frac{5}{6}$. Case (i) is proved. The same arguments can be identically applied to case (ii). Then the lemma is proved.

As a consequence of above lemma and Theorem 2.1, we have

Corollary 2.1. *If $n = 7$ or 8, then M admits a Kähler-Einstein metric with positive scalar curvature.*

In order to complete the proof of main theorem, it remains to consider the case that $n = 5$ or 6. In fact, by the same arguments in the proof of Lemma 2.3, one can have analogous estimates of the integrals $I(\alpha, i)$ in (2.10) (cf. Lemma 2.5 in the proof in the following). Of course, the involved computations are more complicated. Instead, we give an alternative discussion here.

Proposition 2.1. *Let $\{f_i\}$ be a sequence of holomorphic functions on the unit ball $B_1 = \{z \in C^2 | |z| < 1\}$ such that $\lim_{i \to \infty} = f, f \not\equiv 0$. Let $\beta > 0$ be such that the integral $\int_{|z| \leqq 1} |f|^{2\beta} dV$ is finite, then for any $\alpha < \beta$, we have*

$$\int_{|z| \leqq \frac{1}{2}} \frac{dV}{|f|^{2\alpha}} = \lim_{i \to \infty} \int_{|z| \leqq \frac{1}{2}} \frac{dV}{|f_i|^{2\alpha}} \tag{2.22}$$

where dV is the standard volume form on C^2.

Remark. In fact, Lemma 2.3 is a corollary of Proposition 2.1 and some properties of plurianticanonical divisors. We gave a separated proof because it is much simpler and transparent. The above proposition should be a classical result. But

since I could not find its proof in literature to my limited knowledge, I include a sketched proof in Appendix 2. Actually, the proof is based on the modification of the above arguments in that of Lemma 2.3. The key point is how to make induction in general as we did before. The induction, as well as the proof for the following lemma, is completed by means of Newton polyhedrons associated to holomorphic functions (see Appendix 1 for details).

Lemma 2.4. *Let $n = 5$ or 6 and S be a global section in $H^0(M, K_M^{-6})$. Then there is an $\varepsilon > 0$, such that* (i) *if $n = 6$, we have*

$$\int_M \|S\|^{-\frac{2+\varepsilon}{9}} dV_{\tilde{g}} < \infty \tag{2.23}$$

unless the reduced divisor $\{S = 0\}_{\text{red}}$ is an anticanonical divisor and the union of three lines on M intersecting at a common point, where by a line on M, we mean an irreducible curve of degree 1 with respect to the anticanonical line bundle K_M^{-1}. (ii) *if $n = 5$, we also have* (2.23) *unless either $\{S = 0\}$ contains a curve with multiplicity 9 or $\{S = 0\}_{\text{red}}$ is an anticanonical divisor and the union of two lines and a curve of degree 2 intersecting transversally at a common point.*

In order to avoid distracting the readers from the main stream in the proof of our main theorem, we postpone the proof of this lemma in Appendix 1.

Lemma 2.5. *Let $n = 5$ or 6 and $I(\alpha, i)$ be defined as in* (2.10). *Then there are constants $\varepsilon > 0$ and $C_\alpha > 0$ depending on α, where $0 < \alpha < \frac{2}{3}$, such that* (i) *For $\alpha < \frac{2}{3}$, we have*

$$I(\alpha, i) \leqq C_\alpha \tag{2.24}$$

(ii) *For $\alpha \geqq \frac{2}{3}$, we have*

$$I(\alpha, i) \leqq C_\alpha \cdot \lambda_{N-1}(i)^{-\frac{\alpha}{3}} \tag{2.25}$$

where $N = N_6$ as in Lemma 2.2. Note that ε, C_α are independent of i.

Proof. Choose $\varepsilon > 0$ such that 4ε is given in Lemma 2.4. First we assume that $n = 6$. As before, put $D = \{\tilde{S}_N = 0\}$. If D_{red} is not an anticanonical divisor consisting of three lines intersecting at a common point, then by Lemma 2.4(i), for $\alpha \leqq \frac{2}{3} + \varepsilon$

$$\int_M \|\tilde{S}_N\|^{-\frac{2\alpha}{6}} dV_{\tilde{g}} < \infty \tag{2.26}$$

By Proposition 2.1, we have for $\alpha \leqq \frac{2}{3} + \varepsilon$

$$\begin{aligned} \lim_{i \to \infty} I(\alpha, i) &\leqq \lim_{i \to \infty} \int_{M_i} \|\tilde{S}_{\nu N}^i\|^{-\frac{2\alpha}{6}} dV_{\tilde{g}_i} \\ &= \int_M \|\tilde{S}_N\|^{-\frac{2\alpha}{6}} dV_{\tilde{g}} < \infty \end{aligned} \tag{2.27}$$

Hence, there are constants C_α depending on α such that $I(\alpha, i) \leqq C_\alpha$ for all i and $\alpha \leqq \frac{2}{3} + \varepsilon$. Thus we may assume that $\tilde{S}_N = (\tilde{S})^6$, where $\tilde{S}$ is an anticanonical section

which zero divisor is the union of three lines intersecting at a common point. One can easily check that for $\alpha < \frac{2}{3}$,

$$\int_M \|\tilde{S}_N\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} < \infty \tag{2.28}$$

Then (1) follows from (2.28) and Proposition 2.1 as above.

Let p be the intersection point of the three lines in $\{\tilde{S} = 0\}$, then for any open neighborhood U of p and $0 < \alpha \leqq \frac{2}{3} + \varepsilon$,

$$\int_{M \setminus U} \|\tilde{S}_N\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} < \infty \tag{2.29}$$

It is true simply because $\{\tilde{S} = 0\}$ is smooth outside p. On the other hand, by the fact that no four lines of M intersect at a common point and $\tilde{S}_{N-1}$ is linearly independent of $\tilde{S}_N$, one can easily show that if the neighborhood U of p is sufficiently small, then for $\alpha \leqq \frac{2}{3} + \varepsilon$,

$$\int_U \|\tilde{S}_{N-1}\|^{\frac{-2\alpha}{6}} < \infty \tag{2.30}$$

Now applying Proposition 2.1, we have for $\alpha \leqq \frac{2}{3} + \varepsilon$,

$$\begin{aligned}
\lim_{i \to \infty} (\lambda_N^{\alpha/3-1} I(\alpha, i)) &< \lim_{i \to \infty} \int_{M_i} \frac{1}{(\|\tilde{S}^i_{vN-1}\|^2 + \|\tilde{S}^i_{vN}\|^2)^{\frac{\alpha}{6}}} dV_{\tilde{g}_i} \\
&\leqq \lim_{i \to \infty} \left(\int_{U_i} \|\tilde{S}^i_{vN-1}\|^{\frac{-\alpha}{3}} dV_{\tilde{g}_i} + \int_{M_i \setminus U_i} \|\tilde{S}^i_{vN}\|^{\frac{-\alpha}{3}} dV_{\tilde{g}_i} \right) \\
&= \int_U \|\tilde{S}_{N-1}\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} + \int_{M \setminus U} \|\tilde{S}_N\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} < \infty
\end{aligned} \tag{2.31}$$

where U_i are open sets of M_i and converge to U as $i \to \infty$. Thus (i) is proved.

Next we assume that $n = 5$. If neither D contains a curve with multiplicity 9 nor D_{red} is an anticanonical divisor consisting of two lines and a curve of degree 2 intersecting at a common point, then both (i) and (ii) follows from Proposition 2.1 and Lemma 2.4 as before. Therefore, we may assume that $\tilde{S}_N$ is the section listed in Lemma 2.4(ii) as an exceptional case. If D contains a curve with multiplicity 9, then $D = 9L_1 + 3(L_2 + \ldots + L_6)$, where $L_i (1 \leqq i \leqq 6)$ are lines in M satisfying: $L_1 \cdot L_j = 1 (j \geqq 2)$, $L_i \cdot L_j = 0$ for $i, j \geqq 2$. It follows that there are exactly sixteen such divisors of K_M^{-6}. If D does not contain any curve with multiplicity 9, then $D = 6(L_1 + L_2 + E)$, where L_1, L_2 are lines in M and E is a curve of degree 2 satisfying: L_1, L_2, E intersect to each other at a common point. There are exactly 40 such sections. We will call a section described as above the one of special type in $H^0(M, K_M^{-6})$. Take any two different sections S_1', S_2' in $H^0(M, K_M^{-6})$ described as above, by the fact that each point of M can lie in at most two lines, one can easily check the following estimate $\alpha < \frac{3}{4}$,

$$\int_M (\|S_1' S_2'\|^{\frac{-\alpha}{6}}) dV_{\tilde{g}} < \infty \tag{2.32}$$

If $\tilde{S}_{N-1}$ is not a section of special type in $H^0(M, K_M^{-6})$, then by lemma 2.4(ii) and Proposition 2.1, for $\alpha \leqq \frac{2}{3} + \varepsilon$

$$\lim_{i\to\infty} (\lambda_{N-1}(i)^{\frac{\alpha}{3}} I(\alpha, i)) \leqq \lim_{i\to\infty} \int_{M_i} \|\tilde{S}^i_{\nu N-1}\|^{\frac{-\alpha}{3}} dV_{\vartheta_i} = \int_M \|\tilde{S}_{N-1}\|^{\frac{-\alpha}{3}} dV_\vartheta < \infty \tag{2.33}$$

If $\tilde{S}_{N-1}$ is a section of special type, by Proposition 2.1 and (2.32), we have that for $\alpha \leqq \frac{2}{3} + \varepsilon < \frac{3}{4}$

$$\begin{aligned}\lim_{i\to\infty} (\lambda_{N-1}(i)^{\frac{\alpha}{3}} I(\alpha, i)) &\leqq \lim_{i\to\infty} \int_{M_i} (\|\tilde{S}^i_{\nu N-1}\|^2 + \|\tilde{S}^i_{\nu N}\|^2)^{\frac{-\alpha}{6}} dV_{\vartheta_i} \\ &\leqq 2^{\frac{-\alpha}{6}} \lim_{i\to\infty} \int_{M_i} \|\tilde{S}^i_{\nu N-1}\tilde{S}^i_{\nu N}\|^{\frac{-\alpha}{6}} dV_{\vartheta_i} \\ &= 2^{\frac{-\alpha}{6}} \int_M \|\tilde{S}_{N-1}\tilde{S}_N\|^{\frac{-\alpha}{6}} dV_{\vartheta_i} < \infty \end{aligned} \tag{2.34}$$

Now (ii) of this lemma follows from (2.33) and (2.34).

Lemma 2.6. *Suppose that M is one of complex surfaces of form either $CP^2 \# 5\overline{CP^2}$ or $CP^2 \# 6\overline{CP^2}$ with positive first Chern class. Then we have*

$$\sup_{M_i} \varphi_i \leqq -\delta^{-1} \log(\lambda_{N-1}(i)) + C \tag{2.35}$$

where δ, C are constants independent of i.

Proof. By Lemma 2.2, 2.4, there are two constants $\varepsilon > 0$ and $C' > 0$ such that for all i

$$\int_{M_i} e^{-\left(\frac{2}{3}+\varepsilon\right)\left(\varphi_i - \sup_{M_i}\varphi_i\right)} dV_{\vartheta_i} \leqq C' \lambda_{N-1}(i)^{-\frac{2}{9}-\frac{\varepsilon}{3}} \tag{2.36}$$

Using the equation $(1.1)_{t_i}$, we can rewrite (2.36) as

$$\int_{M_i} e^{\left(\frac{2}{3}+\varepsilon\right)\sup_{M_i}\varphi_i + \left(\frac{1}{3}-\varepsilon\right)\varphi_i} dV_{\vartheta_i} \leqq C' e^{\sup_{M_i} f_i} \cdot \lambda_{N-1}(i)^{-\frac{2}{9}-\frac{\varepsilon}{3}} \tag{2.37}$$

where $f_i = f_{t_i}$ are given in $(1.1)_{t_i}$. By the concavity of logarithmic functions, we obtain

$$(2 + 3\varepsilon)\sup_{M_i} \varphi_i \leqq -(1 - 3\varepsilon)\inf_{M_i} \varphi_i - (\tfrac{2}{3} + \varepsilon)\log \lambda_N(i) + C'' \tag{2.38}$$

where C'' is a constant independent of i. On the other hand, it is proved in [T1] (also cf. [T2] for stronger form) that

$$-\inf_{M_i} \varphi_i \leqq 2 \sup_{M_i} \varphi_i + C''' \tag{2.39}$$

where C''' is a constant independent of i. Then (2.35) follows from (2.38) and (2.39) by taking $\delta = (6\varepsilon)^{-1}(\frac{2}{3} + \varepsilon)$ and $C = \varepsilon^{-1}(C'' + C''')$.

Corollary 2.2. *Let M be given as in Lemma 2.5. If $\lambda_{N-1} \neq 0$, then M admits a Kähler-Einstein metric with positive scalar curvature.*

Proof. Since $\lambda_{N-1} \neq 0$, all numbers $-\log \lambda_{N-1}(i)$ are uniformly bounded independent of i. Then by Lemma 2.5, there are uniform C^0-estimates of the solutions φ_{t_i} of $(1.1)_{t_i}$. So the corollary follows from the discussion in §1.

Lemma 2.7. *Let M be as in Lemma 2.5 and $\lambda_{N-1} = 0$. Then there is an $\varepsilon' > 0$, such that for any solution ϕ_i of $(1.1)_{t_i}$, we have*

$$-\inf_{M_i} \phi_i \leqq (2-\varepsilon')\sup_{M_i} + C \tag{2.40}$$

where C is a constant independent of i.

Proof. We define two functionals first considered by Aubin in [Au] (see also [BM], [T1]) as follows,

$$I_i(u) = \int_{M_i} u\left(\omega_{\tilde{g}_i}^2 - \left(\omega_{\ i} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^2\right)$$

$$J_i(u) = \int_0^1 \frac{I(su)}{s}\,ds$$

where the metric $\tilde{g}_i$ is just $\tilde{g}_{t_i}$. Note that $\tilde{g}_i$ converge to a Kähler metric $\tilde{g}$ on M. Then by the proof for Proposition 2.3 in [T1], we have

$$\int_{M_i} (-\phi_i)\omega_{g_i}^2 \leqq I_i(\phi_i) - J_i(\phi_i) \tag{2.41}$$

where g_i is the unique Kähler-Einstein metric on M_i and $\omega_{g_i} = \omega_{\tilde{g}_i} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi_i$.

As in the proof of Lemma 2.2 in [T1], we compute

$$\begin{aligned}(I_i - J_i)(\phi_i) &= \int_{M_i} \frac{\sqrt{-1}}{2\pi}\partial\phi_i \wedge \bar{\partial}\phi_{\ i} \wedge \left(\frac{1}{3}\omega_{\tilde{g}_i} + \frac{2}{3}\omega_{g_i}\right) \\ &\quad + \int_{M_i} \phi_i(\omega_{\tilde{g}_i} - \omega_{g_i})\left(\frac{1}{3}\omega_{\tilde{g}_i} + \frac{2}{3}\omega_{g_i}\right) \\ &= -\frac{2}{3}\int_{M_i} \phi_i\omega_{g_i}^2 + \frac{2}{3}\int_{M_i} \phi_i\omega_{\tilde{g}_i}^2 - \frac{1}{3}\int_{M_i} \frac{\sqrt{-1}}{2\pi}\partial\phi_i \wedge \bar{\partial}\phi_i \wedge \omega_{\tilde{g}_i}\end{aligned} \tag{2.42}$$

It follows from this and (2.41) that

$$\int_{M_i} (-\phi_i)\omega_{\tilde{g}_i}^2 \leqq 2\sup_{M_i} \phi_i - \frac{\sqrt{-1}}{2\pi}\int_{M_i} \partial\phi_i \wedge \bar{\partial}\phi_i \wedge \omega_{\tilde{g}_i} \tag{2.43}$$

Take a smooth point x_∞ on the zero divisor D of the section $\tilde{S}_N$ in M, where $N = N_v$ and $v = 6$. Let x_i be on the divisor $\{\tilde{S}_{vN}^i = 0\}$ in M_i such that $\lim_{i\to\infty} x_i = x_\infty$ as M_i converge to M. Let $\eta > 0$ be small. Then one can choose neighborhoods U_i of x_i in M_i, U_∞ of x_∞ in M and local coordinates (z_i, w_i) at x_i, (z, w) at x_∞ such that $x_i = (0,0) \in U_i = \{|z_i| < \eta, |w_i| < \eta\}$, $x_\infty = (0,0) \in U_\infty = \{|z| < \eta, |w| < \eta\}$,

$\lim_{i\to\infty} z_i = z$, $\lim_{i\to\infty} w_i = w$ and $w_i = \tilde{S}^i_{\nu N}$ in U_i, $w = \tilde{S}_N$ in U_∞. Then for i sufficiently large such that $\lambda_{N-1}(i) < \eta^4$, by the choice of the above (z_i, w_i), one can have the estimate

$$\begin{aligned}
&\frac{\sqrt{-1}}{2\pi}\int_{M_i}\partial\log\left(\sum_{\beta=0}^{N}\|\lambda_\beta(i)\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\wedge\bar{\partial}\log\left(\sum_{\beta=0}^{N}\|\lambda_\beta(i)\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\wedge\omega_{\tilde{g}_i}\\
&\quad\geqq\frac{\sqrt{-1}}{2\pi}\int_{\substack{|x_i|<\eta\\ \sqrt[3]{\lambda_{N-1}(i)}<|w_i|<\sqrt[4]{\lambda_{N-1}(i)}}}\frac{C^{-1}(|w_i|^2-C\lambda_{N-1}(i))}{(|w_i|^2+C|\lambda_{N-1}(i)|^2)^2}dw_i\wedge d\bar{w}_i\wedge dz_i\wedge d\bar{z}_i\\
&\quad\geqq-\varepsilon''\log(\lambda_{N-1}(i))
\end{aligned}$$

where C, $\varepsilon'' > 0$ are some constants independent of i.

On the other hand, by Lemma 2.2, we have

$$\begin{aligned}
&\left|\frac{\sqrt{-1}}{2\pi}\int_{M_i}\left(\partial\phi_i\wedge\bar{\partial}\phi_i-\partial\log\left(\sum_{\beta=0}^{N}\|\lambda_\beta(i)\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\right.\right.\\
&\qquad\left.\left.\wedge\bar{\partial}\log\left(\sum_{\beta=0}^{N}\|\lambda_\beta(i)S^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\wedge\omega_{\tilde{g}_i}\right|\right.\\
&\qquad\leqq\int_{M_i}\left|\phi_i-\sup_{M_i}\phi_i-\log\left(\sum_{\beta=0}^{N}\|\lambda_\beta(i)S^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\right|\\
&\qquad\times\left(\omega_{g_i}+2\omega_{\tilde{g}_i}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{\beta=0}^{N}\|\lambda_\beta(i)S^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\right)\\
&\qquad\leqq C\text{ for some constant independent of }i
\end{aligned}$$

Recall that $-\inf_{M_i}\phi_i$ is dominated by the average $\int_{M_i}(-\phi_i)\omega^2_{g_i}$ (cf. [T1]). Then it follows from (2.41) and the above inequalities

$$-\inf_{M_i}\phi\leqq 2\sup_{M_i}\phi_i+\varepsilon''\log(\lambda_{N-1}(i))+C' \tag{2.44}$$

where C' is a constant independent of i. Now (2.40) follows from (2.44), Lemma 2.5.

Now our main theorem follows from Corollary 2.1, 2.2, Theorem 2.1 and Lemma 2.5(i) and Lemma 2.7.

3. An application of Gromov's compactness theorem

In this section, we will apply Gromov's compactness theorem ([GLP], [GW]) and Uhlenbeck's curvature estimate for Yang-Mills equation to studying the degeneration of Kähler-Einstein metrics for compact complex surfaces in $\mathfrak{I}_n(5\leqq n\leqq 8)$. It is the first step towards the proof of Theorem 2.2 (strong partial C^0-estimate). The more general version of the result in this section, i.e., Proposition 3.2, is stated in [An] and [Na]. But for reader's convenience and our own sake, we include a complete and independent proof here for our special case of Kähler-Einstein metrics.

Let $\{(M_i, g_i)\}$ be a fixed sequence of compact Kähler surfaces in $\mathfrak{I}_n$ with positive first Chern class and Kähler-Einstein metric g_i, where $5 \leqq n \leqq 8$. We normalize each metric g_i such that $\mathrm{Ric}(g_i) = \omega_{g_i}$.

For each Kähler-Einstein manifold (M_i, g_i), one can define a tensor $T(i)$ on M_i, which measures the deviation of Kähler manifold (M_i, g_i) from being of constant holomorphic sectional curvature. In local coordinates (z_1, z_2) of M_i, define

$$T(i)_{\alpha\bar{\beta}\gamma\bar{\delta}} = R(i)_{\alpha\bar{\beta}\gamma\bar{\delta}} - \tfrac{1}{3}(g_{i\alpha\bar{\beta}} g_{i\gamma\bar{\delta}} + g_{i\alpha\bar{\delta}} g_{i\gamma\bar{\beta}}) . \tag{3.1}$$

$$(1 \leqq \alpha, \beta, \gamma, \delta \leqq 2)$$

where $R(i)$ denotes the bisectional curvature tensor of the metric g_i.

A straightforward computation as in [Y2] shows the following equality for each (M_i, g_i) with M_i in $\mathfrak{I}_n$ $(3 \leqq n \leqq 8)$,

$$\int_{M_i} \|T(i)\|^2_{g_i} dV_{g_i} = \tfrac{4}{3}(3C_2(M_i) - C_1^2(M_i)) = \tfrac{8}{3}(9 - n) . \tag{3.2}$$

where $\|T(i)\|_{g_i}$ is the norm of the tensor $T(i)$ with respect to g_i, that is in local coordinates (z_1, z_2),

$$\|T(i)\|^2_{g_i}(x) = g_i^{\alpha\bar{\alpha}'} g_i^{\beta\bar{\beta}'} g_i^{\gamma\bar{\gamma}'} g_i^{\delta'\delta} T(i)_{\alpha\bar{\beta}\gamma\bar{\delta}} \overline{T(i)_{\alpha'\bar{\beta}'\gamma'\bar{\delta}'}}(x) . \tag{3.3}$$

One may also see [Ban] for reference of (3.2), too.

In particular, it implies that the L^2-integral of $\|R(i)\|_{g_i}$ is uniformly bounded from above by a uniform constant.

Lemma 3.1. *Let (M_i, g_i) be a Kähler-Einstein surface given as above. Then there are uniform constants C', C'' such that for any f in $C^1(M_i, R)$*

$$C'\left(\int_{M_i} |f|^4 dV_{g_i}\right)^{\frac{1}{2}} - C'' \int_{M_i} |f|^2 dV_{g_i} \leqq \int_{M_i} |\nabla f|^2 dV_{g_i} \tag{3.4}$$

where ∇f denotes the gradient of f.

Proof. Since $\mathrm{Ric}(g_i) = \omega_{g_i}$ and $\mathrm{Vol}_{g_i} M_i = 9 - n$ is a constant, the lemma follows from a combination of results in C. Croke [Cr] and P. Li [Li].

The following lemma is essentially due to *K*. Uhlenbeck [Uh2].

Lemma 3.2. *Let N be the integer $\left[\dfrac{50}{(C')^2}\right] + 1$, where C' is the Sobolev constant given in* (3.4), *$[a]$ denotes the integer part of the real number a. Then there is a universal constant $C \geqq 0$, such that for any $r \in (0, 1)$ and any Kähler-Einstein surface (M_i, g_i) given as above, there are finite many points $x_{i1}^r, \ldots, x_{in}^r$ in M_i such that*

$$\|R(i)\|_{g_i}(x) \leqq \frac{C}{r^2}\left(\int_{B^{g_i}_{\frac{r}{4}}(x, g_i)} \|R(i)\|^2_{g_i}(x) dV_{g_i}\right)^{\frac{1}{2}} \quad \text{for any} \quad x \in M_i \setminus \bigcup_{\beta=1}^{N} B_r(x_{i\beta}^r, g_i) . \tag{3.5}$$

where $B_r(x_{i\beta}^r, g_i)$ is the geodesic ball with radius r and center at $x_{i\beta}^r$ and $\|R(i)\|_{g_i}$ is the norm of $R(i)$ with respect to g_i.

Proof. A straightforward computation shows

$$-\Delta_{g_i}(\|R(i)\|_{g_i}) \leqq \|R(i)\|_{g_i} + \mu(\|R(i)\|_{g_i})^2 \tag{3.6}$$

where Δ_{g_i} is the laplacian of g_i and μ is a positive constant independent of i, which actual value is not important to us. Define

$$E_i = \left\{ x \in M_i \mid \int_{B_{\frac{r}{4}}(x, g_i)} \|R(i)\|_{g_i}^2 dV_{g_i} \geqq \varepsilon \right\} \tag{3.7}$$

Then by (3.2) and the well-known covering lemma, E_i can be covered by N geodesic balls of radius $\frac{r}{2}$. Take $x_{i1}^r, \ldots, x_{iN}^r$ to be the centers of these balls. Then for any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$,

$$\int_{B_{\frac{r}{4}}(x, g_i)} \|R(i)\|_{g_i}^2 dV_{g_i} \geqq \varepsilon \tag{3.8}$$

Let $\eta: R_+^1 \to R_+^1 = \{t \in R^1 \mid t \geqq 0\}$ be a cut-off function satisfying $\eta \equiv 1$ for $t \leqq 1$; $\eta \equiv 0$ for $t \geqq 2$ and $|\eta'(t)| \leqq 1$.

For any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$, denote by $\rho_x(\cdot)$ the distance function on M_i from x.

Put $f = \|R(i)\|_{g_i}$. Multiplying $\eta^2\left(\frac{8\rho_x}{r}\right)f$ on both sides of (3.6) and then integrating by parts, one obtains

$$\int_{M-i} |\nabla(\eta f)|^2 dV_{g_i} \leqq \int_{M_i} \eta^2 f^2 dV_{g_i} + \int_{M_i} |\nabla\eta|^2 f^2 dV_{g_i} + \int_{M_i} \eta^2 f^4 dV_{g_i} \tag{3.9}$$

By Lemma 3.1 and Hölder inequality,

$$C'\left(\int_{M_i} |\eta f|^4 dV_{g_i}\right)^{\frac{1}{2}} - C'' \int_{M_i} |\eta f|^2 dV_{g_i} \leqq \int_{M_i} \left(\eta^2 + \frac{64|\eta'|^2}{r^2}\right)|f|^2 dV_{g_i} + \left(\int_{M_i} |\eta f|^4 dV_{g_i}\right)^{\frac{1}{2}} \left(\int_{B_{\frac{r}{4}}(x, g_i)} |f|^2 dV_{g_i}\right)^{\frac{1}{2}} \tag{3.10}$$

Therefore, for some universal constant $C \geqq 0$, we have

$$\left(\int_{B_{\frac{r}{4}}(x, g_i)} |f|^4 dV_{g_i}\right)^{\frac{1}{2}} \leqq \frac{C}{r^2(C' - \sqrt{\varepsilon})} \int_{B_{\frac{r}{4}}(x, g_i)} |f|^2 dV_{g_i} \tag{3.11}$$

Similarly, by multiplying $\eta^2 f^3$ on both sides of (3.6) and processing as above, we have

$$\left(\int_{B_{\frac{r}{16}}(x, g_i)} |f|^8 dV_{g_i}\right)^{\frac{1}{2}} \leqq \frac{C}{r^2(C' - \sqrt{\varepsilon})} \int_{B_{\frac{r}{8}}(x, g_i)} |f|^4 dV_{g_i} \tag{3.12}$$

Combining (3.11) and (3.12), and letting $\varepsilon \leqq \frac{1}{4}(C')^2$,

$$\left(\int_{B_{\frac{r}{16}}(x, g_i)} |f|^8 dV_{g_i}\right)^{\frac{1}{8}} \leqq \frac{C}{r^{3/2}} \left(\int_{B_{\frac{r}{4}}(x, g_i)} |f|^2 dV_{g_i}\right)^{\frac{1}{2}} \tag{3.13}$$

Then (3.3) follows from Moser's iteration as in the proof of Theorem 8.17 in [GT].

We further observe that we may take the set $\{x_{i1}^{\frac{r}{4}}, \ldots, x_{iN}^{\frac{r}{4}}\}$ contained in the union of the balls $B_r(x_{i\beta}^r, g_i)$. Let $\{r_j\}_{j\geq 1}$ be a decreasing sequence of positive numbers such that $r_1 \leqq \frac{1}{4}$, $r_j \leqq \frac{r_{j-1}}{4}$ and if we write $x_{i\beta}^j$ as $x_{i\beta}^{r_j}$ and define

$$\Omega_i^j = M_i \bigcup_{\beta=1}^N B_{2r_j}(x_{i\beta}^j, g_i) \tag{3.14}$$

Then

$$\bar{\Omega}_i^j \subseteq \Omega_i^{j+1}\left(\frac{r_{j+1}}{8}\right), \quad \text{and} \quad \bigcup_{j\geq 1}^{j} \Omega_i^j = M_i \setminus \{x_{i1}, \ldots, x_{iN}\}$$

where $x_{i\beta} = \lim_{j\to\infty} x_{i\beta}^j$ for any $1 \leqq \beta \leqq N$,
$\Omega_i^{j+1}(\varepsilon) = \{x \in \Omega_i^{j+1} | \mathrm{dist}_{g_i}(x, \partial\Omega_i^{j+1}) > \varepsilon\}$.

The following proposition is essentially a special case of the famous Gromov's compactness theorem (cf. [GP], [GW]).

Proposition 3.1. *Let $\{(X_i, h_i)\}$ be a sequence of n-dimensional Kähler-Einstein manifolds (maybe noncompact) and Ω_i be a sequence of domains in X_i with boundary $\partial\Omega_i$ for each $i \geqq 1$. Suppose that all i,*

(i) *The norm $\|R(h_i)\|_{h_i}(x)$ of the bisectional curvatures $R(h_i)$ are uniformly bounded for x in Ω_i.*
(ii) $\mathrm{InjRad}(x) \geqq C$ *for $x \in \Omega_i$ and some uniform constant C.*
(iii) $0 \leqq C' \leqq \mathrm{Vol}_{h_i}(\Omega_i) \leqq C''$ *for some uniform constants C', C''.*

Then given any $\varepsilon > 0$, there is a subsequence $\{\Omega_{i_k}(\varepsilon), h_{i_k}\}_{k\geq 1}$ of Kähler-Einstein manifolds $\{\Omega_i(\varepsilon), h_i\}_{i\geq 1}$, where $\Omega_i(\varepsilon) = \{x \in \Omega_i | \mathrm{dist}_{h_i}(x, \partial\Omega_i) > \varepsilon\}$, and a Kähler-Einstein manifold $(\Omega_\infty(\varepsilon), h_\infty)$ such that for the compact subset $K \subset \Omega_\infty(\varepsilon)$, there is an $\varepsilon' > \varepsilon$ such that for k sufficiently large, there are diffeomorphisms ϕ_{i_k} of $\Omega_{i_k}(\varepsilon')$ into $\Omega_\infty(\varepsilon)$ satisfying
(1) $K \subset \phi_{i_k}(\Omega_{i_k}(\varepsilon'))$ *for any $k \geqq 1$.*
(2) $(\phi_{i_k}^{-1})^* h_{i_k}$ *converge uniformly to h_∞ on K.*
(3) $(\phi_{i_k})_* \cdot J_{i_k} \cdot (\phi_{i_k}^{-1})_*$ *converge uniformly to J_∞ on K, where J_{i_k}, J_∞ are the almost complex structures of Ω_i, $\Omega_\infty(\varepsilon)$, respectively.*

Proof. By some standard computations and the assumption that (X_i, h_i) are Kähler-Einstein manifolds, the bisectional curvature tensor $R(h_i)$ satisfies a quasi-linear elliptic system. The assumption (i), (ii), and (iii) imply that the Sobolev inequalities hold on $\Omega_i(\varepsilon)$ with uniform Sobolev constant. It follows from some well-known elliptic estimates (cf. [GT], [Uh1]) that

$$\|D^l R(h_i)\|_{h_i}(x) \leqq C(l), \quad l = 1, 2, \ldots, \infty . \tag{3.16}$$

where $D^l R(h_i)$ denotes the l^{th} covariant derivative of $R(h_i)$ on Ω_i and $C(l)$ are uniform constants depending only on l. Then by Gromov's compactness theorem ([GP], [GW]), there are a subsequence $\{(\Omega_{i_k}(\varepsilon), h_{i_k})\}$ and a Riemannian manifold $(\Omega_\infty(\varepsilon), h_\infty)$ such that the above (1) and (2) hold. Let K be any compact subset in $\Omega_\infty(\varepsilon)$ and ϕ_{i_k} defined as in the statement of this proposition. For the almost complex structure J_{i_k} on Ω_{i_k}, it is clear that $(\phi_{i_k})_* \cdot J_{i_k} \cdot (\phi_{i_k}^{-1})_*$ is an almost complex

on K. By taking subsequence of $\{i_k\}$, we may assume that $(\phi_{i_k})_* \cdot J_{i_k} \cdot (\phi_{i_k}^{-1})$ converge on K. Since K is arbitrary, we obtain an almost complex structure J_∞ on $\Omega_\infty(\varepsilon)$. It is easy to check that this J_∞ is integrable and h_∞ is a Kähler-Einstein metric with respect to this J_∞.

Combining this proposition with lemma 3.2, we have the following corollary.

Lemma 3.3. *Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein surfaces in Theorem 3.1. By taking a subsequence of $\{(M_i, g_i)\}$, we may assume that $(M_i \setminus \{x_{i\beta}\}_{1 \leq \beta \leq N}, g_i)$ converge to a Kähler-Einstein manifold (M_∞, g_∞) in the following sense: for any compact subset $K \subset M_\infty$, there is a $r > 0$ such that there are diffeomorphisms ϕ_i from $M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}, g_i)$ into M_∞ with K in the images and satisfying:*

(1) *$(\phi_i^{-1})^* g_i$ converge to g_∞ uniformly on K.*

(2) *$\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to J_∞ uniformly on K, where J_i, J_∞ are the almost complex structures of M_i, M_∞, respectively.*

Moreover, the limit M_∞ has only finite many connected components and the curvature tensor $R(g_\infty)$ of g_∞ is L^2-bounded by a universal constant.

Proof. For any $j \geqq 1$, by Lemma 3.2, the curvature tensor $R(i)$ of g_i are uniformly bounded on the domains Ω_i^j in (3.14), and $\mathrm{Vol}_{g_i}(\Omega_i^j)$ uniformly approximte to $(9 - n)\pi^2$. Since $\mathrm{Ric}(g_i) = \omega_{g_i}$ for all i, the diameters $\mathrm{diam}(M_i, g_i)$ are bounded from above by $\sqrt{3}\pi$. By Volume Comparison Theorem [Bi], we have for any $0 < r < \sqrt{3}\pi$ and $x \in M_i$,

$$\mathrm{Vol}(B_r(x, g_i)) \geqq cr^4 \tag{3.17}$$

where c is a uniformly constant. Thus by the estimate on injectivity radius in [CGT], we prove that those assumptions (i)–(iii) in Proposition 3.1 are satisfied by (Ω_i^j, g_i), $i, j \geqq 1$. Then by Proposition 3.1, we have a sequence of open Kähler-Einstein surfaces $(\Omega_\infty^j, g_\infty^j)$. By the properties (1)–(3) in that proposition, we can naturally identify Ω_∞^j with a domain in Ω_∞^{j+1} such that the restriction of g_∞^{j+1} to Ω_∞^j is just g_∞^j. Thus we glue $\{(\Omega_\infty^j, g_\infty^j)\}_{j \geqq 1}$ together to obtain the required (M_∞, g_∞) with properties (1) and (2) as stated in this lemma. It is clear by Fatou's lemma that $R(g_\infty)$ is L^2-bounded by the universal constant for the L^2-bounded of $R(g_i)$. The finiteness of the connected components of M_∞ follows from Lemma 3.4 we will prove in the following.

Let ρ_i be the distance function on $M_i \times M_i$ induced by the metric g_i, and ρ_∞ be the limit of ρ_i. Note that to make $\rho_\infty = \lim \rho_i$ meaningful, we may need to take the subsequence of $\{i\}$. Obviously, this function ρ_∞ is a Lipschitz function on $M_\infty \times M_\infty$. Also for each β between 1 and N, the function $\rho(x_{i\beta})$ converge to a Lipschitz function $\rho_{\infty\beta}$. According to [GLP], one may attach finite many points $x_{\infty 1}, \ldots, x_{\infty N}$ to M_∞ such that $M_\infty \cup \{x_{\infty 1}, \ldots, x_{\infty N}\}$ becomes a complete length space with length function ρ_∞ extending that ρ_∞ on $M_\infty \times M_\infty$, $\rho_\infty(x_{\infty\beta}, \cdot) = \rho_\infty(\cdot, x_{\infty\beta}) = \rho_{\infty\beta}$.

Lemma 3.4. *For any $r > 0$, put $E_\beta(r) = \{x \in M_\infty \mid \rho_{\infty\beta}(x) < r\}$, then there is a constant L independent of r such that the number of the connected components in $E_\beta(r)$ is less than L for any $1 \leqq \beta \leqq N$.*

Proof. By the construction of M_∞, it is easy to show that any point in $E_\beta(r)$ can be connected to a point in $E_\beta(r')$ by a path within $E_\beta(r)$ for any $r' > 0$. Thus it suffices to prove the lemma for sufficiently small r. Choose $r_0 > 0$ such that for $r \leqq r_0$, $E_\beta(r) \cap E'_\beta(r) = \varnothing$ for $\beta \neq \beta'$. By Volume Comparison Theorem [Bi] and positivity of the Ricci curvatures of (M_i, g_i), by taking limit of $B_r(x_{i\beta}, g_i)$, $B_{\frac{r}{4}}(x_{i\beta}, g_i)$ etc., as i goes to infinity, we have for $1 \leqq \beta \leqq N$,

$$\mathrm{Vol}_{g_\infty}(E_\beta(r)) \leqq Cr^4 \tag{3.18}$$

$$\mathrm{Vol}_{g_\infty}(B_{\frac{r}{4}}(x, g_\infty)) \geqq C^{-1}r^4 \quad \text{for} \quad x \in \partial E_\beta\left(\frac{r}{2}\right) \tag{3.19}$$

where C is some constant independent of r.

Put $L = [C^2] + 2$. We claim that the number of connected components in $E_\beta(r)$ is less than L. In fact, if not, by taking r smaller, we may have $y_1, \ldots, y_L$ in different components of $\partial E_\beta\left(\frac{r}{2}\right)$ such that $B_{\frac{r}{4}}(y_j, g_\infty) \cap B_{\frac{r}{4}}(y_{j'}, g_\infty) = \varnothing$ for $j \neq j'$ and $B_{\frac{r}{4}}(y_i, g_\infty) \subset E_\beta(r)$. Thus by (3.18) and (3.19), we have

$$C^{-1}r^4 L \leqq \sum_{j=1}^{L} \mathrm{Vol}_{g_\infty}(B_{\frac{r}{4}}(y_j, g_\infty)) \leqq \mathrm{Vol}_{g_\infty}(E_\beta(r)) \leqq Cr^4$$

i.e. $L \leqq C^2$. A contradiction. Therefore, our claim is true and the lemma is proved.

Lemma 3.5. *There is a decreasing positive function $\varepsilon(r)$, satisfying $\lim_{r\to 0}\varepsilon(r) = 0$, such that for any point x in M_∞, we have*

$$\|R(g_\infty)\|(x) \leqq \frac{\varepsilon(r(x))}{r^2(x)}$$

where $r(x) = \min_{1\leqq j\leqq N}\{\rho_\infty(x_{\infty j}, x)\}$.

Proof. It simply follows from Lemma 3.2 by taking limit on i and using Lemma 3.3.

Denote by $\Delta\left(\frac{1}{k}, k\right)$ the truncated ball $\left\{z \middle| \frac{1}{k} < |z| < k\right\}$ in euclidean space C^2. Put $\Delta_r^* = \bigcup_{r\geqq r'>0}\Delta(\frac{1}{2}r', 2r')$, then Δ_r^* is the punctured ball in C^2 with radius r. Also denote by g_F the standard euclidean metric on C^2.

Lemma 3.6. *Let E be one of connected components in $\bigcup_{\beta=1}^N E_\beta(r_0)$, where r_0 is chosen as in the proof of Lemma 3.4 such that $E_\beta(r_0) \cap E_{\beta'}(r_0) = \varnothing$ for $\beta \neq \beta'$. Then there are a $\tilde{r} > 0$ and a diffeomorphism ϕ from $\Delta_{\tilde{r}}^*$ into the universal covering $\tilde{E}$ of $E \cap \bigcup_{\beta=1}^N E_\beta(\tilde{r})$ such that the covering map $\pi_E: \tilde{E} \to E$ is finite and for $r \geqq \tilde{r}$,*

$$\max_{\Delta_r^*} |(\pi_E \circ \phi)^* g_\infty - g_F|_{g_r} \leqq \varepsilon_1(r) \tag{3.21}$$

where $\varepsilon_1(r)$ is a decreasing function of r with $\lim_{r\to\infty}\varepsilon_1(r) = 0$.

Proof. For any integer $k \geqq 2$, $kr \leqq r_0$, we define an open manifold $D(r, k)$ with

Kähler metric $g(r, k)$,

$$D(r, k) = E \cap \bigcup_{\beta=1}^{N} \left(E_\beta(kr) \setminus E_\beta\left(\frac{r}{k}\right) \right).$$

$$g(r, k) = \frac{1}{r^2} g_\infty |_{D(r,k)}$$

Then by Lemma 3.5,

$$\| R(g(r, k)) \|_{g(r,k)} \leqq k^2 \varepsilon(kr) \tag{3.22}$$

Claim 1. For any fixed integer $k > 0$ and any sequence $\{r(i)\}_{i \geqq 1}$ with $kr(i) \leqq r_0$ and $\lim r(i) = 0$, there is a subsequence $\{i_j\}$ of $\{i\}$ such that Kähler-Einstein manifolds $(D(r(i_j), k), g(r(i_j), k))$ converge to a flat Kähler manifold $D_{\infty,k}$. Here the meaning of convergence is as that in Lemma 3.3.

This is simply a consequence of Proposition 3.1 and (3.22) and the fact that each $(D(r, k), g(r, k))$ is Kähler-Einstein.

For simplicity, we assume that the subsequence $\{r(i_j)\}$ is just $\{r(i)\}$ in the above claim. By the diagonal method and taking subsequence of $\{r(i)\}$ if necessary, we may assume that for any $k' \geqq 2$, $(D(r(i), k'), g(r(i), k'))$ converge to a flat Kähler manifold $D_{\infty,k'}$. We can naturally identify $D_{\infty,k'}$ as an open set of $D_{\infty,k''}$ if $k' < k''$. Thus each manifold $D_{\infty,k}$ in claim 1 is contained as open subset in a flat Kähler manifold $D_\infty = \bigcup_{k' \geqq 2} D_{\infty,k'}$. As before, the distance function $\frac{1}{r(i)^2}\rho_\infty$ of the dilated metric $\frac{1}{r(i)^2} g(r(i), k)$ converge to the distance function ρ_F of the flat metric g_F on D_∞.

Let E be in $E_\beta(r_0)$, then $\frac{1}{r(i)^2}\rho_\beta$ also converge to a Lipschitz function, formally denoted by $\rho_F(o, \cdot)$. One may think o as an attached point to D_∞. Note that $D_{\infty,k} = \left\{ z \in D_\infty \middle| \frac{1}{k} < \rho_F(0, z) < k \right\}$. Let $\tilde{D}_\infty$ be the universal covering of D_∞. Since $\tilde{D}_\infty$ is flat and simple-connected, we may assume that $\tilde{D}_\infty$ is an open subset in C^2.

Claim 2. The fundamental group $\pi_1(D_\infty)$ is finite. In fact, the number of elements in $\pi_1(D_\infty)$ is bounded by a uniform constant.

By the construction of D_∞, it is easy to find a point y in D_∞ with $\rho_F(0, y) = 1$ and a geodesic ray γ in D_∞ such that $\gamma(1) = y$ and $\rho_F(0, \gamma(t)) = t$. For any $R > 0$, define a modified geodesic ball of radius $R > 0$ by

$$B_R^m(\gamma(3R), g_F) = \{ z \in D_\infty | \rho_F(\gamma(3R), z) < R \text{ and } \exists \text{ a unique geodesic jointing } z \text{ to } \gamma(3R) \} \tag{3.22}$$

Then the closure of $B_R^m(\gamma(3R), g_F)$ is the limit of the balls $B_R\left(y_{iR}, \frac{1}{r(i)^2} g_\infty\right)$ in $E \subset E_\beta(r_0)$, where all y_{iR} lie on a geodesic $\gamma_0 \subset E$, whose dilations by $r(i)^{-1}$ converge to the previous ray γ as i goes to infinity. It follows from Volume

Comparison Theorem [Bi] that

$$\mathrm{Vol}_{g_F}(B_R^m(\gamma(3R), g_F)) = \mathrm{Vol}_{g_F}(\overline{B_R^m(\gamma(3R), g_F)})$$
$$= \lim_{i\to\infty} \mathrm{Vol}_{g_\infty}\left(B_R^m\left(y_{iR}, \frac{1}{r(i)^2} g_\infty\right)\right) \geqq CR^4$$

where C is a uniform constant. Let $\tilde{\gamma}$ be a lifting of γ to $\tilde{D}_\infty$ such that $\pi(\tilde{\gamma}(1)) = \gamma(1) = y$, where $\pi: \tilde{D}_\infty \to D_\infty$ is natural projection. Also $\pi(\tilde{\gamma}(3R)) = \gamma(3R)$. Since $B_R^m(\gamma(3R), g_F)$ is contractible by definition, there is an open subset $\tilde{B}_R \subset \tilde{D}_\infty$ such that $\pi|_{\tilde{B}_R}$ is an isometry from $\tilde{B}_R$ onto $B_R^m(\gamma(3R), g_F)$. In particular, $\mathrm{Vol}_{g_F}(\tilde{B}_R) \geqq CR^4$ with C given above. Any element σ in $\pi_1(D_\infty)$ can be considered as a deck transformation on $\tilde{D}_\infty$. By definition of $\tilde{B}_R$, we have $\sigma(\tilde{B}_R) \cap \tilde{B}_R = \varnothing$. Denote by $B_{R'}(\tilde{\gamma}(1))$ the standard ball in C^2 with center at $\tilde{\gamma}(1)$. Then

$$(36\pi)^2 R^4 = \mathrm{Vol}_{g_F}(B_{6R}(\tilde{\gamma}(1))) \geqq \sum_{\mathrm{dist}(\sigma(\tilde{\gamma}(1)), \tilde{\gamma}(1)) < R} \mathrm{Vol}_{g_F}(\sigma(\tilde{B}_R))$$
$$\geqq CR^4 \cdot \#\{\sigma \in \pi_1(D_\infty) | \mathrm{dist}(\sigma(\tilde{\gamma}(1)), \tilde{\gamma}(1)) < R\}$$

By letting R go to infinity, we conclude

$$\#\{\sigma \in \pi_1(D_\infty)\} \leqq \frac{(36\pi)^2}{C} < \infty .$$

Claim 2 is proved.

Any element in $\pi_1(D_\infty)$ is an isometry of the open subset $\tilde{D}_\infty$ in C^2. Thus $\pi_1(D_\infty)$ can be considered as a subgroup in the linear automorphism group of C^2. Since $\pi_1(D_\infty)$ is finite, we may further assume that $\pi_1(D_\infty)$ is in the unitary group $U(2)$.

Claim 3. The Kähler manifold $\tilde{D}_\infty \subset C^2$ coincides with $C^2 \backslash \{o\}$. Moreover, the pullback $\pi^* \rho_F(o, \cdot)$ coincides with the euclidean distance function from the origin in C^2.

We adopt the notations in the proof of Claim 2. Let $q = \lim_{t\to+0} \tilde{\gamma}(t)$. Let $\overline{\tilde{D}_\infty}$ be the closure of $\tilde{D}_\infty$ in C^2, and $p \in \overline{\tilde{D}_\infty} \backslash \tilde{D}_\infty$. Then there is a sequence $\{p_j\} \subset \tilde{D}_\infty$ such that $p = \lim p_j$ in C^2. It is clear that $\{\pi(p_j)\}$ has no limit point in D_∞, so $\lim_{j\to\infty} \rho_F(o, \pi(p_j)) = 0$. Each $\pi(p_j)$ can be connected to a $\gamma(t_j)$ by a path γ_j' with length $l(\gamma_j')$ and $\lim_{j\to\infty} l(\gamma_j') = 0$. Thus for each j, there is another lifting $\tilde{p}_j$ of $\pi(p_j)$ in $\tilde{D}_\infty$ such that $\lim_{j\to\infty} \tilde{p}_j = q$ in C^2. For each j, there is a $\sigma_j \in \pi_1(D_\infty)$ such that $p_j = \sigma_j(\tilde{p}_j)$. By taking a subsequence of $\{j\}$ if necessary, we may assume that all σ_j are the same, denoted by σ. Then $p = \sigma(q)$. Thus we proved that $\tilde{D}_\infty = C^2 \backslash \pi_1(D_\infty) \cdot q$. Let o be the origin of C^2, then $o \notin \tilde{D}_\infty$ since $\pi_1(D_\infty) \subset U(2)$ acts on $\tilde{D}_\infty$ freely. It follows that $q = 0$ and $\tilde{D}_\infty = C^2 \backslash \{o\}$.

Claim 4. For any $\varepsilon \in (0, 1)$, there is a $r_\varepsilon > 0$ such that for any $r > r_\varepsilon$, there is a diffeomorphism ϕ_r from $\Delta(\frac{1}{2}r, 2r)$ into $\pi_E^{-1}(D(r, 2))$ with its image containing $\pi_E^{-1}(D(r, 2 - \varepsilon))$ and

$$\max\left\{\|\phi_r^* \pi_E^* g_\infty - g_F\|_{g_r}(x) | x \in \Delta\left(\frac{1}{2}r, 2r\right)\right\} \leqq \varepsilon \tag{3.23}$$

where g_F is induced by the euclidean metric on C^2.

We prove it by contradiction. Suppose that the claim is not true, then there is a sequence $\{r(i)\}$ with $\lim r(i) = 0$ such that for any $r(i)$ no diffeomorphism with the above properties exists. But by previous three claims, we can easily find a subsequence of $\{r(i)\}$, for simplicity, say $\{r(i)\}$ itself, such that $\left(D(2, r(i)), \frac{1}{r(i)^2} g_\infty\right)$ converge to $\Delta(\frac{1}{2}, 2)/\Gamma$ in $D_\infty = C^2 \setminus \{0\}/\Gamma$ for some finite group $\Gamma \in U(2)$ with $\# \Gamma \leqq C$, where C is a uniform constant given in Claim 2. Since the estimate (3.23) is invariant under scaling, by the definition of the convergence given in Proposition 3.1, we may have the diffeomorphisms ϕ_i from $\Delta(\frac{1}{2}r(i), 2r(i))$ into $\pi_E^{-1}(D(r(i), 2))$ for i large satisfying (3.23) above. A contradiction. We proved this claim.

This above claim implies immediately that there is a decreasing function $\varepsilon'(r)$ on r with $\lim_{r \to \infty} \varepsilon'(r) = 0$, such that for any $r < r_{\frac{1}{2}}$, we can replace ε by $\varepsilon'(r)$ in (3.23) of Claim 4.

It remains to glue all ϕ_r together to obtain the required local diffeomorphism ϕ in the statement of our lemma. Put $r_1 = \frac{1}{2}\tilde{r}$, $r_i = \frac{1}{2}r_{i-1}$ for $i \geqq 2$, where $\tilde{r}$ is sufficiently small. Let $\phi_i = \phi_{r_i}$ be the diffeomorphism from $\Delta(\frac{1}{2}r_i, 2r_i)$ into $\pi_E^{-1}(D(r_i, 2))$ given by Claim 4. Then for any $i \geqq 2$, the composition $\phi_{i-1}^{-1} \circ \phi_i$ is a diffeomorphism from $(\pi_E \circ \phi_i)^{-1}(D(r_i, 2) \cap (r_{i-1}, 2))$ onto $(\pi_E \circ \phi_{i-1})^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))$ and satisfies

$$\sup\{\|(\phi_{i-1}^{-1} \circ \phi_i)^* g_F - g_F\|_{g_r}(x) | x \in (\pi \circ \phi_i)^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))\} \leqq 4\varepsilon'(r_{i-1}) \tag{3.24}$$

$$\sup\{\|(\phi_i^{-1} \circ \phi_{i-1})^* g_F - g_F\|_{g_r}(x) | x \in (\pi_E \circ \phi_{i-1})^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))\} \leqq 4\varepsilon'(r_{i-1}) \tag{3.25}$$

By (3.24) and (3.25) and letting $\tilde{r}$ small enough, one can easily modify $\phi_{i-1}^{-1} \circ \phi_i$ to be a smooth diffeomorphism ψ_i from $(\pi_E \circ \phi_i)^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))$ into $\Delta_{\tilde{r}}^*$ such that

$$\psi_i = \begin{cases} \phi_{i-1}^{-1} \circ \phi_i & \text{in } (\pi_E \circ \phi_i)^{-1}((D(r_{i-1}, 2) \cap D(\frac{3}{10}r_{i-1}, 2)) \\ \mathrm{Id} & \text{in } (\pi_E \circ \phi_i)^{-1}(D(r_i, 2) \cap D(\frac{5}{3}r_{i-1}, 2)) \end{cases}$$

and the estimates

$$\sup\{\|\psi_i^* g_F - g_F\|_{g_r} | x \in (\pi_E \cdot \phi_i)^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))\} \leqq 400\varepsilon(r_{i-1}) \tag{3.26}$$

Now we define a diffeomorphism $\phi: \Delta_{\tilde{r}}^* \to \tilde{E}$ by

$$\phi|_{\Delta(\frac{3}{5}r_1, 2r_1)} = \phi_1, \quad \phi|_{\Delta(\frac{3}{5}r_i, \frac{5}{3}r_i)} = \phi_i (i \geqq 2)$$

$$\phi|_{\Delta(\frac{3}{5}r_{i+1}, \frac{5}{3}r_i)} = \phi_i \circ \psi_{i+1} \quad \text{for} \quad i \geqq 1 .$$

Then ϕ satisfies (3.21) for some decreasing function $\varepsilon(r)$ with $\lim_{r \to 0} \varepsilon_1(r) = 0$. The finiteness of π_E follows directly from Claim 2. The lemma is proved.

By this lemma, we can compactify M_∞ topologically by adding a point $x_{\infty\beta}$ to $E_\beta(\tilde{r})$ for each β between 1 and N. Denote by $\bar{M}_\infty$ the compactification of M_∞. Then $\bar{M}_\infty$ has the following properties: for any $x_{\infty\beta}$, there is a neighborhood U_β of $x_{\infty\beta}$ in $\bar{M}_\infty$ such that any connected component $U_{\beta j} (1 \leqq j \leqq l_\beta)$ of $U_\beta \cap M_\infty$ is covered by a smooth manifold $\tilde{U}_{\beta j}$ with the covering group $\Gamma_{\beta j}$ isomorphic to a finite group in $U(2)$, and $\tilde{U}_{\beta j}$ is diffeomorphic to a punctured ball $\Delta_{\tilde{r}}^*$ in C^2. Let

$\phi_{\beta j}$ be the diffeomorphism from $\Delta_{\tilde{r}}^*$ onto $\tilde{U}_{\beta j}$ and $\pi_{\beta j}$ be the covering map from $\tilde{U}_{\beta j}$ onto $U_{\beta j}$. Then by Lemma 3.6, the pull-back metric $\phi_{\beta j}^* \circ \pi_{\beta j}^*(g_\infty)$ extends to a C^0-metric on the ball $\Delta_{\tilde{r}}$ with the estimate (3.21). Note that the metric $\phi_{\beta j}^* \circ \pi_{\beta j}^*(g_\infty)$ is an Einstein one outside the origin.

Such a $\bar{M}_\infty$ with the metric g_∞ is called a generalized topological orbifold with C^0-metric g_∞. The previous discussions in this section yields.

Proposition 3.2. *Let $\{(M_i, g_i)\}$ be a sequence of compact Kähler-Einstein surface in $\mathfrak{I}_n(5 \leqq n \leqq 8)$ as given at the beginning of this section. Then by taking a subsequence if necessary, we may assume that (M_i, g_i) converge to an open Kähler-Einstein surface (M_∞, g_∞) with $M_\infty = \bar{M}_\infty \backslash \{x_{\infty\beta}\}_{1 \leqq \beta \leqq N}$ in the sense of Lemma 3.3, where $\bar{M}_\infty$ is a generalized topological orbifold such that g_∞ can be extended to be a C^0-metric on $\bar{M}_\infty$ described as above.*

The differential structure on M_∞ can be extended to $\bar{M}_\infty$ and the extension may not be unique. But there is at most one with which the metric g_∞ on M_∞ can be extended smoothly to $\bar{M}_\infty$. In next section, we will prove that there is such an extension by using Uhlenbeck's theory of removing singularities of Yang-Mills connections. We would like to point out that Proposition 3.2 also holds for a sequence of real 4-dimension Einstein manifolds as claimed and proved in [An] and [Na]. We refer readers to these papers.

4. Removing isolated singularities of Kähler-Einstein metrics

In [Uh1], K. Uhlenbeck invented a beautiful theory of removing isolated singularities of Yang-Mill connections on real 4-dimensional manifolds. The purpose of this section is to apply this theory of Uhlenbeck to the Kähler-Einstein metric g_∞ on M_∞ constructed in the last section and prove that g_∞ can be smoothly extended to the generalized topological orbifold $\bar{M}_\infty$ with some differential structure (cf. Proposition 3.2 for details). The latter $\bar{M}_\infty$ is considered as a compactification of M_∞ and the complement $\bar{M}_\infty \backslash M_\infty$ consists of finitely many points $\{x_{\infty\beta}\}_{1 \leqq \beta \leqq N}$.

Let $U_{\beta j}$ be any connected component in $U_\beta \cap M_\infty (1 \leqq \beta \leqq N, 1 \leqq j \leqq l_\beta)$, where U_β is a small neighborhood of $x_{\infty\beta}$ in $\bar{M}_\infty$. Recall that each $U_{\beta j}$ is covered by $\Delta_{\tilde{r}}^*$ in C^2 with the covering group $\Gamma_{\beta j}$ isomorphic to a finite group in $U(2)$ and $\pi_{\beta j}^* g_\infty$ extends to a C^0-metric on the ball $\Delta_{\tilde{r}}$ with the estimate (3.21). The smooth extension of g_∞ to $\bar{M}_\infty$ is local in nature. Therefore, we may

Fix β and j $(1 \leqq \beta \leqq N, 1 \leqq j \leqq l_\beta)$ and denote $\pi_{\beta j}^* g_\infty$ by g for simplicity. We need to construct a homeomorphism ψ of $\Delta_{\tilde{r}}$ into itself, such that the restriction of ψ to $\Delta_{\tilde{r}}^*$ has its image in $\Delta_{\tilde{r}}^*$ and is C^∞-smooth and $\psi^* g$ extends smoothly across the origin in $\Delta_{\tilde{r}}^*$.

The first step towards this goal is to prove the boundedness of the curvature tensor $R(g)$. The proof here is identical to that for Yang-Mill connections in [Uh1] with some modifications. However, for reader's convenience, we include a sketched proof here. We will just consider $\Delta_{\tilde{r}}^*$ as a real 4-dimensional manifold for being, where $\tilde{r}$ is given in Lemma 3.6. In the proof of Lemma 3.6, we observe that by the definition of the metric g, we may choose the diffeomorphism $\phi = \phi_{\beta j}$ properly

such that for $\tilde{r}$ sufficiently small, the following estimate hold,

$$\begin{cases} \|dg_{ij}\|_{gF}(x) \leqq \dfrac{\varepsilon_1(r(x))}{r(x)}, & x \in \Delta_{\tilde{r}}^*,\ 1 \leqq i, j, \leqq 4 \\ \left\|d\left(\dfrac{\partial g_{ij}}{\partial x_k}\right)\right\|_{gF}(x) \leqq \dfrac{\varepsilon_1(r(x))}{r(x)^2}, & x \in \Delta_{\tilde{r}}^*,\ 1 \leqq i, j, k \leqq 4 \\ \left\|d\left(\dfrac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right)\right\|_{gF}(x) \leqq \dfrac{\varepsilon(r(x))}{r(x)^3}, & x \in \Delta_{\tilde{r}}^*,\ 1 \leqq i, j, k, l \leqq 4 \end{cases} \tag{4.1}$$

where d is the exterior differential on $C^2 = R^4$, $\|\cdot\|_{gF}$ is the norm on T^1R^4 with respect to the euclidean metric g_F and $g_{ij} = g\left(\dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j}\right)$ for the standard coordinates (x_1, x_2, x_3, x_4) for R^4.

Let $\tilde{A}$ be the connection form uniquely associated to the metric g on $\Delta_{\tilde{r}}^*$, that is, $\tilde{D} = d + \tilde{A}$ is the covariant derivative with respect to g. Clearly, we can regard $\tilde{A}$ as a function in $C^{1,\alpha}(\Delta_{\tilde{r}}^*, so(4) \times R^4)$ for $\alpha \in (0, 1)$.

The following lemma is essentially Theorem 2.8 in [Uh1].

Lemma 4.1. *Let $\tilde{r}$ be sufficiently small. Then there is a gauge transformation u in $C^\infty(\Delta(r, 2r), so(4))$ satisfying: if $D = e^{-u} \cdot \tilde{D} \cdot e^u = d + A$, then $d^*A = 0$ on $\overline{\Delta(r, 2r)}$, $d_\psi^* A_\psi = 0$ on $\partial\Delta(r, 2r)$ and $\int_{\Delta(r, 2r)} A(\nabla_F r(x))dV_g = 0$, where d^*, d_ψ^* are the adjoint operators of the exterior differentials on $\Delta(r, 2r)$ or $\partial\Delta(r, 2r)$ with respect to g, respectively, and ∇_F denotes the standard gradient, dV_g is the volume form of g. Moreover, we have*

$$\sup_{\Delta(r, 2r)} (\|A\|_g(x)) \leqq \frac{\varepsilon_2(r)}{r} \tag{4.2}$$

where $\varepsilon_2(r)$ is a decreasing function on r with $\lim_{r \to 0} \varepsilon_2(r) = 0$.

Proof. As in [Uh1], the proof follows from an application of Implicit Function Theorem. For reader's convenience, we sketch a proof here.

By scaling, we may take $r = 1$ and $\bar{\Omega} = \overline{\Delta(1, 2)}$ with the scaled metric $\dfrac{1}{r^2}g$ is sufficiently close to the flat metric g_F on $\bar{\Omega}$ if r is small enough. Let $C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*)$ be the collection of $C^{2,\alpha}$-smooth covariant symmetric tensors on $\bar{\Omega}$. Then for any $h \in C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*)$ sufficiently close to the zero tensor, we have a new metric $g_F + h = g_h$, consequently, it induces a unique $so(4)$-connection A_h on $\bar{\Omega}$. As we have pointed out in the above (4.1), the difference of the scaled metric $\dfrac{1}{r^2}g$ from g_F is small in $C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*)$ whenever r is small. We define operators

$$\begin{aligned} Q: C_0^{2,\alpha}(\bar{\Omega}, so(4)) \times C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*) &\to C^{0,\alpha}(\bar{\Omega}, so(4)) \times C_0^{0,\alpha}(\partial\bar{\Omega}, so(4)) \\ (u, h) &\to (d_h^*(e^{-u}de^u + e^{-u}A_h e^u), d_{h\psi}^*(e^{-u}d_\psi e^u + e^{-u}A_{h\psi}e^u)) \\ f: C_0^{2,\alpha}(\bar{\Omega}, so(4)) \times C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*) &\to so(4) \\ (u, h) &\to \int_{\bar{\Omega}} (e^{-u}de^u + e^{-u}A_h e^u)(\nabla r)dV_{g_h} \end{aligned}$$

where $0 < \alpha < 1$, the subspace $C_0^{k,\alpha}(\cdot,\cdot)$ consists of all functions orthogonal to constant ones ($k = 0, 2$), and d_h^*, $d_{h\psi}^*$ are the adjoint operators of d, d_ψ with respect to the metric g_h, respectively.

To prove the lemma, it suffices to find a u such that $(Q,f)\left(u, \frac{1}{r^2}g - g_F\right) = 0$. One can easily check that the partial derivative of the operator (Q,f) with respect to u is an isomorphism at the point $(0,0)$, so the lemma follows from Implicit Function Theorem.

Lemma 4.2. *Let A be the connection for given in Lemma* 4.1, *then for r small, we have*

$$\sup_{\Delta(r,2r)} \|A\|_g(x) \leqq Cr \sup_{\Delta(r,2r)} \|R_A\|_g(x) \tag{4.3}$$

$$\int_{\Delta(r,2r)} \|A\|_g^2(x)dV_g \leqq Cr^2 \int_{\Delta(r,2r)} \|R_A\|_h^2(x)dV_g \tag{4.4}$$

Proof. Both (4.3) and (4.4) are invariant under scaling. So it suffices to prove the lemma on $\Omega = \Delta(1,2)$ with the scaling metric $\frac{1}{r^2}g$. By Lemma 3.6, the metric $\frac{1}{r^2}g$ converge uniformly to g_F on $\bar{\Omega}$ as r goes to zero. Thus by the proof of Corollary 2.9 in [Uh1], we conclude that

$$\lambda(r) = \inf\left\{ \frac{\int_{\Omega} \|df\|^2_{\frac{1}{r^2}g} dV_g}{\int_{\Omega} \|f\|^2_{\frac{1}{r^2}g} dV_g} \;\middle|\; f \in C^2(\bar{\Omega}, T^*_{R^4} \otimes so(4)),\ d^*f = 0,\ d^*_\psi f\Big|_{\partial\Omega} = 0,\ \int_{\Omega} f(\nabla r(x))dV_{\frac{1}{r^2}g} = 0 \right\}$$

has a uniform lower bound λ independent of r. By the equation $dA + [A, A] = DA = R_A$, where D is the covariant derivative associated to A, we have

$$\lambda \int_{\Delta(1,2)} \|A\|_g^2 dV_g \leqq \int_{\Delta(1,2)} \|dA\|_g^2 dV_g \leqq 2 \int_{\Delta(1,2)} \|R_A\|_g^2 dV_g + 2 \int_{\Delta(1,2)} \|[A, A]\|_g^2 dV_g$$

By the estimate on $\|A\|_g$ in Lemma 4.1, the last integral is bounded by $C\varepsilon_2(r)\int_{\Delta(1,2)} \|A\|_g^2 dV_g$ for some constant C independent of r. Then (4.4) follows when r is sufficiently small.

On the other hand, by Lemma 4.1 and the equation $dA + [A, A] = R_A$, we have

$$\|dA\|_g(x) \leqq \|R_A\|_g(x) + C\varepsilon_2(r)\|A\|_g(x) \tag{4.5}$$

where C is some uniform constant. Then (4.3) follows easily from (4.4), (4.5) and the fact that $d^*A = 0$ and the scaled metric $\frac{1}{r^2}g$ is close to the flat metric g_F on $\bar{\Omega}$ if r is sufficiently small.

Lemma 4.3. *Given any $1 > \delta > 0$, there is a $r(\delta) > 0$, $r(\delta) < \tilde{r}$ such that*

$$\|R(g)\|_g(x) \leqq \frac{1}{r(x)^\delta} \quad \text{on} \quad \Delta^*_{r(\delta)} \tag{4.6}$$

Proof. The proof is essentially same as that of Proposition 4.7 in [Uh1]. So we just sketch the proof here. Choose a small $r < \tilde{r}$, put $r_1 = \frac{r}{2}, r_2 = \frac{r_1}{2}, \ldots, r_i = \frac{r_{i-1}}{2}, \ldots$. Let A_i be the connection on $\Delta(r_i, r_{i-1})$ given by Lemma 4.1. Then $d^*_\psi A_{i\psi}|_{\partial\Delta(r_i, r_{i-1})} = 0$, $d^*_\psi A_{i-1\psi}|_{\partial\Delta(r_i, r_{i-1})} = 0$, it follows that the restrictions $A_{i\psi}$ and $A_{i-1\psi}$ to $\partial\Delta_{r_{i-1}}$ are distinct by a constant gauge on $\partial\Delta_{r_{i-1}}$. So we may assume that $A_{i\psi}|_{\partial\Delta r_{i-1}} = A_{i-1\psi}|_{\partial\Delta r_{i-1}}$. Put $\Omega_i = \Delta(r_i, r_{i-1})$, then we have

$$\begin{aligned}
\int_{\Omega_i} \|R_{A_i}\|^2_g dV_h &= \int_{\Omega_i} \langle dA_i + [A_i, A_i], R_{A_i}\rangle_g dV_g \\
&= \int_{\Omega_i} \langle D_i A_i - [A_i, A_i], R_{A_i}\rangle_g dV_g \qquad (4.7)_i \\
&= -\int_{\Omega_i} \langle [A_i, A_i], R_{A_i}\rangle_g dV_g - \int_{\Omega_i} \langle A_i, D_i^* R_{A_i}\rangle_g dV_g \\
&\quad - \int_{S_i} \langle A_{i\psi}, (R_{A_i})_{r\psi}\rangle_g d\sigma_g + \int_{S_{i-1}} \langle A_{i\psi}, (R_{A_i})_{r\psi}\rangle_g d\sigma_g
\end{aligned}$$

where $D_i = d + A_i$, $S_i = \partial\Delta_{r_i}$ and $d\sigma$ is the induced volume form on S_i, S_{i-1}, etc. Because the metric g is Einstein and A_i is equivalent to $\tilde{A}$ by gauge transformation, we have $D^* R_{A_i} = 0$. On the other hand, by (4.1) and Lemma 4.1, one can easily show that $\|A_i\|_g(x) \leqq \frac{C\varepsilon_2(r(x))}{r(x)}$, $\|R_{A_i}\|_g(x) \leqq \frac{C\varepsilon_2(r(x))}{r(x)^2}$ for some constant C independent of i and x in Ω_i. Thus by summing equations $(4.7)_i$ over $i \geqq 1$ and observing that $(R_{A_i})_{r\psi} = (E_{A_{i-1}})_{r\psi}$ on S_i, we obtain

$$\begin{aligned}
\int_{\Delta_r} \|R(g)\|^2_g dV_g &= \sum_{i=1}^{\infty} \int_{\Omega_i} \|R(g)\|^2_g dV_g \\
&= \sum_{i=1}^{\infty} \int_{\Omega_i} \|R_{A_i}\|^2_g dV_g \qquad (4.8) \\
&= -\sum_{i=1}^{\infty} \int_{\Omega_i} \langle R_{A_i}, [A_i, A_i]\rangle_g dV_g + \int_{\partial\Delta_r} \langle A_{1\psi}, (R_{A_i})r\psi\rangle_g dV_g
\end{aligned}$$

By Lemma 4.2 and the previous estimate on $\|R_A\|_g(x)$, we have

$$\begin{aligned}
\left| \int_{\Omega_i} \langle R_{A_i}, [A_i, A_i]\rangle_g dV_g \right| &\leqq C' \sup_{\Omega_i}(\|R_{A_i}\|_g(x)) \int_{\Omega_i} \|A_i\|^2_g dV_g \\
&\leqq C\varepsilon_2(r_i) \int_{\Omega_i} \|R_{A_i}\|^2_g dV_g \qquad (4.9)
\end{aligned}$$

where C, C' are some constants independent of i.

By Lemma 3.6, one can see that $\left(\partial\Delta_r, \frac{1}{r^2}g\right)$ converge uniformly on (S^3, g_F), i.e. the unit sphere with the standard metric. Then by the proof of corollary 2.6 in [Uh1], one can find a decreasing function $\varepsilon'(r)$ on r with $\lim_{r\to 0}\varepsilon'(r) = 0$ such that

$$\int_{\partial\Delta_r} \|A_{1\psi}\|_g^2 dV_g \leqq (2-\varepsilon'(r))^{-2} r^2 \int_{\partial\Delta_r} \|(F_{A_1})_{\psi\psi}\|_g^2 dV_g \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we have

$$\int_{\Delta_r} \|R(g)\|_g^2 dV_g \leqq (2-\varepsilon'(r))^{-1}(1+C\varepsilon_2(r))r\left(\int_{\partial\Delta_r} \|(R_{A_1})_{\psi\psi}\|_g^2 dV_g\right)^{1/2}$$

$$\left(\int_{\partial\Delta_r} \|(R_{A_1})_{r\psi}\|_g^2 dV_g\right)^{1/2} \leqq \frac{(2+\varepsilon'(r))^{-1}(1+C\varepsilon_2(r))r}{2} \int_{\partial\Delta_r} \|R(g)\|_g^2 dV_g$$

$$\leqq \frac{r}{4}\left(1+\frac{\delta}{2}\right)\int_{\partial\Delta_r} \|R(g)\|_g^2 dV_g.$$

whenever $r(\delta)$ is sufficiently small and $r \leqq r(\delta)$.

Then it is standard to conclude from above inequality that

$$\int_{\Delta_r} \|R(g)\|_g^2 dV_g \leqq r^{4-\frac{\delta}{4}} \quad \text{for } r \leqq r(\delta) \tag{4.11}$$

The estimate (4.6) follows from (4.11) and the fact that g is Einstein and close to the flat metric on $\Delta_{\tilde{r}}$.

With help of Lemma 4.3, we can now regularize the extension of the metric g across the origin in $\Delta_{\tilde{r}} \subset C^2$.

Lemma 4.4. *Let g, $\Delta_{\tilde{r}}$, g_F have the meanings as above. Then if $\tilde{r}$ is sufficiently small, there is a self-diffeomorphism ψ of $\Delta_{\tilde{r}}^*$ such that ψ extends to be a homeomorphism of $\Delta_{\tilde{r}}$ and*

$$\|\psi^* g - g_F\|_{gF}(x) \leqq r(x)^{\frac{3}{2}} \quad x \in \Delta_{\tilde{r}}^* \tag{4.12}$$

$$\|d(\psi^* g)\|_{gF}(x) \leqq r(x)^{\frac{1}{2}} \quad x \in \Delta_{\tilde{r}}^* \tag{4.13}$$

Proof. We first construct ψ_r with properties analogous to (4.12) and (4.13) in the annulus $\Delta((\frac{3}{2}-\varepsilon)r, (\frac{3}{2}+\varepsilon)r)$ for some small $\varepsilon > 0$ independent of r. By scaling, we may construct ψ on $\Delta(\frac{3}{2}-\varepsilon, \frac{3}{2}+\varepsilon) \subset \Omega = \Delta(1,2)$ with metric $\frac{1}{r^2}g$, still denoted by g for simplicity. Then by previous lemma, if $r < \tilde{r}$ is small, we have

$$\sup_{\Omega}(\|R(g)\|_g(x)) \leqq r^{\frac{7}{4}} \tag{4.14}$$

and also $\sup_{\Omega}\{\|R(g)\|_g(x)\} \leqq \varepsilon'$, where ε' can be very small if we want.

For any point $x \in \partial\Delta_{\frac{3}{2}}$, using the harmonic coordinates constructed in [Jo], one can find a diffeomorphism ψ_x from the euclidean ball $B_{\frac{1}{4}}(x)$ into Ω such that

$\psi_x(B^1_{\frac{1}{4}}(x))$ is very close to $B^1_{\frac{1}{4}}(x)$ in Hausdorff distance and

$$\sup\{\|\psi_x^* g - g_F\|_{gF}(y), \quad \|d\psi_x^* g\|_{gF}(y) | y \in B^1_{\frac{1}{4}}(x)\} \leqq Cr^{\frac{7}{4}} \tag{4.15}$$

where C is a constant independent of r and x.

Our ψ_r is obtained by gluing finitely many $\psi_{x_i}(x_i \in \partial\Delta^3_{\frac{3}{2}})$. We just sketch this gluing process here. Let (y_1, y_2, y_3, y_4) be the euclidean coordinates, then $\partial\Delta^3_{\frac{3}{2}} = \left\{(y_1, \ldots, y_4)\} | \sqrt{\sum_{i=1}^4 |y_i|^2} = \frac{3}{2}\right\}$. For any fixed a, b with $a^2 + b^2 < \frac{5}{4}$, we choose finitely many points $x_1, \ldots, x_N$ on the circle $S(a, b) \subset \partial\Delta^3_{\frac{3}{2}}$ consisting of all points (y_1, y_2, y_3, y_4) with $(y_3, y_4) = (a, b)$ given above, such that $B^1_{\frac{1}{4}}(x_i) \cap B^1_{\frac{1}{4}}(x_j) = \varnothing$ if $|i - j| > 1$ and the union of $B^1_{\frac{1}{4}}(x_i)(1 \leqq i \leqq N)$ contains the neighborhood $B^1_{\frac{1}{8}}(S(a, b))$ of $S(a, b)$. Note that the number N of $\{x_i\}$ is bounded independent of (y_3, y_4) and r. Now we glue $\psi_{x_1}, \ldots, \psi_{x_N}$ together as in the proof of Lemma 3.6 and obtain a diffeomorphism $\psi_{(a,b)}$ from $B^1_{\frac{1}{8}}(S(a, b))$ into Ω such that the estimate (4.15) holds for this diffeomorphism on $B^1_{\frac{1}{8}}(S(a, b))$. Note that the constant C may be different, but still independent of r. Next, fix $b < \frac{5}{4}$, choose $a_1, \ldots, a_{N'}$ such that $a_1 = -\sqrt{\frac{9}{4} - b^2} < a_2 < \ldots < a_{N'} = \sqrt{\frac{9}{4} - b^2}$ and $B^1_{\frac{1}{4}}(0, 0, a_1, b) \cap B^1_{\frac{1}{8}}(S(a_2, b)) = \varnothing$ for $j \geqq 3$; $B^1_{\frac{1}{8}}(S(a_i, b)) \cap B^1_{\frac{1}{8}}(S(a_j, b)) = \varnothing$ for $|i - j| \geqq 2$; $B^1_{\frac{1}{8}}(S(a_j, b)) \cap B^1_{\frac{1}{4}}(0, 0, a_{N'}, b) = \varnothing$ for $j \leqq N - 2$, also the union of them covers the neighborhood $B_{\frac{1}{16}}(S(b))$ of $S(b) = \{(y_1, y_2, y_3, y_4) \in \partial\Delta^3_{\frac{3}{2}} | y_4 = b\}$. Then we glue $\psi_{(0,0,a_1,b)}$, $\psi_{(0,0,a_{N'},b)}$ and those $\psi_{(a_i,b)}(2 \leqq i \leqq N - 1)$ construct in step one to obtain a diffeomorphism ψ_b from $B_{\frac{1}{16}}(S(b))$ into Ω such that (4.15) holds for ψ_b on $B_{\frac{1}{16}}(s(b))$. Finally, we choose finitely many points $b_1, \ldots, b_{N''}$ in the interval $[-\frac{3}{2}, \frac{3}{2}]$ and repeat the above gluing process for $B^1_{\frac{1}{4}}((0, 0, 0, -\frac{3}{2}))$, $B^1_{\frac{1}{4}}((0, 0, 0, \frac{3}{2}))$ and $B_{\frac{1}{16}}(b_i)$ for $2 \leqq i \leqq N'' - 1$. We then have the diffeomorphism ψ_r from $\Delta(\frac{47}{32}, \frac{49}{32})$ into Ω. By scaling, we may consider ψ_r as a diffeomorphism from $\Delta(\frac{47}{32}r, \frac{49}{32}r)$ into Ω such that the image contains $\Delta((\frac{47}{32} + \tilde{\varepsilon}(r))r, (\frac{49}{32} - \tilde{\varepsilon}(r))r)$ in ω and

$$\sup\left\{\frac{1}{r}\|\psi_r^* g - g_F\|_{gF}(x), \|d\psi_r^* g\|_{gF}(x) | x \in \Delta\left(\frac{47}{32}r, \frac{49}{32}r\right)\right\} \leqq Cr^{\frac{3}{4}} \tag{4.16}$$

where $\tilde{\varepsilon}(r)$ is a decreasing function with $\lim_{r \to 0} \tilde{\varepsilon}(r) = 0$.

Now let $r_1 = \tilde{r}$, $r_{i+1} = \frac{24}{25} r_i$ for $i \geqq 1$. We have constructed diffeomorphisms ψ_i from $\Delta(\frac{47}{49}r_i, r_i)$ into Ω with properties described above. Then the required ψ is obtained by gluing these ψ_i properly as in the proof of Lemma 3.6. The estimates (4.12) and (4.13) follows from (4.16) if $\tilde{r}$ is sufficiently small.

Lemma 4.5. *Let g, Δ_r be as in Lemma 4.4. Then there is a diffeomorphism ψ from Δ_r^* into Δ_r^* such that $\psi^* g$ extends to be a C^∞-metric on Δ_r. Moreover, if J is the almost complex structure on Δ_r^* such that g is Kähler with respect to J, then*

$(\psi^{-1})_* \circ J \circ \psi_*$ extends to be an integrable almost complex structure on $\Delta_{\tilde{r}}$ such that $\psi_* g$ is Kähler-Einstein with respect to it.

Proof. By Lemma (4.4), we may assume

$$\|g - g_F\|_{gF}(x) \leqq r(x)^{\frac{3}{2}}, \quad \|dg\|_{gF}(x) \leqq r(x)^{\frac{1}{2}} \quad \text{on } \Delta_{\tilde{r}}^* \tag{4.17}$$

Then g is a $C^{1,\frac{1}{2}}$-metric on $\Delta_{\tilde{r}}$. Let $x_1, \ldots, x_4$ be the euclidean coordinate functions on $\Delta_{\tilde{r}}$ and

$$|\Delta_g x_i|(x) \leqq Cr(x)^{1/2}, \quad x \in \Delta_{\tilde{r}} \tag{4.18}$$

where Δ_g is the Laplacian of the metric g and C is a constant independent of x, $\tilde{r}$. Solving Dirichlet problems for k_i,

$$\begin{cases} \Delta_g k_i = -\Delta_g x_i & \text{on } \Delta_{\tilde{r}} \\ k_i|_{\partial\Delta_{\tilde{r}}} = \delta_i \end{cases} \tag{4.19}$$

where δ_i are constants of order $O(\tilde{r}^{\frac{1}{2}})$ such that $k_i(0) = 0$. By the standard elliptic theory [GT], we have $C^{2,\frac{1}{2}}$-solutions k_i of (4.19) such that $\sup_{\Delta_{\tilde{r}}}(\|dk_i\|_{gF}) = O(\tilde{r}^{\frac{1}{2}})$. It follows that $(l_1, \ldots, l_4)$, where $l_i = x_i - k_i$, is a harmonic coordinate system in $\Delta_{\tilde{r}}$ with respect to the metric g when $\tilde{r}$ is sufficiently small. Let $\psi : \Delta_{\tilde{r}} \to \Delta_{2\tilde{r}}$ be the diffeomorphism by mapping $(x_1, \ldots, x_4)$ to $(l_1(x), \ldots, l_4(x))$ and $\{g_{ij}\}$ be the tensor representing the metric $\psi^* g$. Then $\{g_{ij}\}$ are $C^{1,\frac{1}{2}}$-smooth and as in [KT], the Einstein condition on g implies

$$-\frac{1}{2}\sum_{r,s} g^{r,s}\frac{\partial^2 g_{ij}}{\partial l_r \partial l_s} + \text{lower order term} = -g_{ij} \text{ on } \Delta_{\tilde{r}} \tag{4.20}$$

By elliptic regularity theory [GT], we conclude that $\{g_{ij}\}$ are C^∞-smooth. In fact, $\{g_{ij}\}$ are real analytic. Now it is clear that $\psi_*^{-1} \circ J \circ \psi_*$ is extendable and $\psi^* g$ is a Kähler-Einstein metric with respect to the extended complex structure on $\Delta_{\tilde{r}}$.

Recall that $\bar{M}_\infty$ is obtained by adding finitely many points to the limit (M_∞, g_∞) of the sequence of Kähler-Einstein manifolds $\{(M_i, g_i)\}$ in Lemma 3.3. Summarizing the above discussions and using Proposition 3.2, we have actually proved that the compactification $\bar{M}_\infty$ of (M_∞, g_∞) has the properties: for any added point $x_\infty \in \bar{M}_\infty \setminus M_\infty$, there is a neighbourhood U of x_∞ in $\bar{M}_\infty$ such that any connected component U_j of $U \cup M_\infty$ is covered by a punctured ball $\Delta_{\tilde{r}}^*$ in C^2 with the covering group isomorphic to a finite group in $U(2)$. Moreover, if $\pi_j : \Delta_{\tilde{r}}^* \to U_j$ is the covering map, then $\pi_j^* g_\infty$ extends to be a Kähler-Einstein metric on $\Delta_{\tilde{r}}$ in C^2 with respect to the standard complex structure. Therefore, we have

Proposition 4.2. *Let $\{(M_i, g_i)\}$ be the sequence of compact Kähler-Einstein manifolds given at beginning of this section. Then by taking a subsequence if necessary, we may assume that (M_i, g_i) converge to a Kähler-Einstein manifold $(M_\infty \setminus \mathrm{Sing}(M_\infty), g_\infty)$ in the sense of Lemma 3.3, where (M_∞, g_∞) is a connected Kähler-Einstein orbifold (maybe reducible) and $\mathrm{Sing}(M_\infty)$ is the finite set of singular points of M_∞.*

Here a complex orbifold is defined in the general sense as described before this proposition. Moreover, from now on, we will say that the sequence of Kähler-Einstein manifolds converge to a Kähler-Einstein orbifold (M_∞, g_∞) if the conclusion in the above proposition is true for $\{(M_i, g_i)\}$.

5. Application of L^2-estimate for $\bar{\partial}$-operators on Kähler-Einstein orbifolds

Let $\{(M_i, g_i)\}_{i\geq 1}$ be a sequence of Kähler-Einstein surfaces in $\mathfrak{I}_n$ with $\mathrm{Ric}(g_i) = \omega_{g_i}$. In previous two sections, it is proved that some subsequence of $\{(M_i, g_i)\}_{i\geq 1}$ converge to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Proposition 4.2. In this section, we will apply L^2-estimate for $\bar{\partial}$-operators to studying the properties of this limiting orbifold M_∞. We will prove that the plurianticanonical group $H^0(M_i, K_{M_i}^{-m})$ converge to $H^0(M_\infty, K_{M_\infty}^{-m})$ for any integer $m > 0$ as (M_i, g_i) converge to (M_∞, g_∞) and M_∞ is an irreducible normal surface with only rational double points and some special Hirzebruch-Jung singularities as singular points (cf. [BPV]).

We first recall the definition of a line bundle on M_∞ (cf. [Bai]).

Definition 5.1. A line bundle on the complex orbifold M_∞ is a line bundle L on the regular part $M_\infty \setminus \mathrm{Sing}(M_\infty)$ such that for each local uniformization $\pi_p: \tilde{U}_p \to M_\infty$ of a singular point p, the pull-back $\pi_p^* L$ on $\tilde{U}_p \setminus \pi_p^{-1}(p)$ can be extended to the whole $\tilde{U}_p$.

On the Kähler-Einstein orbifold M_∞, we have natural line bundles in the sence of Definition 5.1 such as pluricanonical line bundles $K_{M_\infty}^m$ and plurianticanonical line bundles $K_{M_\infty}^{-m} (m \in Z_+)$. A global section of $K_{M_\infty}^m$ is an element in $H^0(M_\infty \setminus \mathrm{Sing}(M_\infty), K_{M_\infty}^m)$. Let $H^0(M_\infty, K_{M_\infty}^m)$ denote the linear space of all these global sections of $K_{M_\infty}^m$. Note that the metric g_∞ induces natural hermitian orbifold metrics h_∞^m on $K_{M_\infty}^{-1}$.

Lemma 5.1. *Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein surfaces given at the beginning of this section and S^i be a global holomorphic section in $H^0(M_i, K_{M_i}^{-m})$ with $\int_{M_i} \|S^i\|_{g_i}^2 dV_{g_i} = 1$, where m is a fixed positive integer. Then there is a subsequence $\{i_k\}$ of $\{i\}$ such that the sections S^{i_k} converge to a global holomorphic section S^∞ in $H^0(M_\infty, K_{M_\infty}^{-m})$. In particular, if $\{S_\beta^i\}_{0\leq\beta\leq N_m}$ is an orthonormal basis of $H^0(M_i, K_{M_i}^{-m})$ with respect to the induced inner product by g_i, then by taking a subsequence, we may assume that $\{S_\beta^i\}_{0\leq\beta\leq N_m}$ converge to an orthonormal basis of a subspace in $H^0(M_\infty, K_{M_\infty}^{-m})$, where $N_m + 1 = \dim_{\mathbf{C}} H^0(M_i, K_{M_i}^{-m})$.*

Remark. Before we prove this lemma, we should justify the meaning of the convergence of $\{S^i\}$ in the above lemma since these sections are no longer on a same Kähler manifold. Recall that Lemma 3.3 says: for any compact subset $K \subset M_\infty \setminus \mathrm{Sing}(M_\infty)$, there are diffeomorphisms ϕ_i from compact subsets $K_i \subset M_i$ onto K such that $(\phi_i^{-1})^* g_i$ and $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to g_∞ and J_∞ on K, respectively. Now with ϕ_i as above, we can push the sections S^i down to the sections $\phi_{i*}(S^i)$ of $\otimes^m(\Lambda^2(T_c M \otimes \overline{T_c M}))$ on K. The convergence in Lemma 5.1 means that for any compact subset K of $M_\infty \setminus \mathrm{Sing}(M_\infty)$ and ϕ_i as above, the sections $\phi_{ik*}(S^{i_k})$ converge to a section S^∞ of $K_{M_\infty}^{-1}$ on K in C^∞-topology. Note that the limit S^∞ is automatically holomorphic.

Proof of Lemma 5.1. Let Δ_i be the laplacian of the metric g_i, then by a direct computation, we have

$$\Delta_i(\|S^i\|_{g_i}^2)(x) = \|D_i S^i\|_{g_i}^2(x) - 2m\|S^i\|_{g_i}^2(x) \tag{5.1}$$

where D_i is the covariant derivative with respct to g_i. Since $\int_{M_i} \|S^i\|^2_{g_i}(x)dV_{g_i} = 1$, by Lemma 3.1 and applying Moser's iteration to (5.1), there is a constant $C(m)$ depending only on m such that

$$\sup_{M_i}(\|S^i\|^2_{g_i}(x)) \leqq C(m) \tag{5.2}$$

Let K be a compact subset in $M_\infty \setminus \mathrm{Sing}(M_\infty)$ and ϕ_i be diffeomorphism from K_i onto K as in the above remark. To prove the lemma, it suffices to show (*): for any integer $l > 0$, the l^{th} convariant derivatives of $\phi_{i*}(S^i)$ with respect to g_∞ are bounded in K by a constant C_l' depending only on l and K. There is a $r > 0$, depending only on K, such that for any point x in K_i, the geodesic ball $B_r(x, g_i)$ is uniformly biholomorphic to an open subset in C^2. On each $B_r(x, g_i)$, the section S_i is represented by a holomorphic function $f_{i,x}$. By (5.1), the function $f_{i,x}$ are uniformly bounded. Therefore, by the well-known Cauchy integral formula, one can easily prove that at x the l^{th} covariant derivative of S^i are uniformly bounded by a constant depending only on l, K. It follows (*) since $(\phi_i^{-1})^* g_i$ uniformly converge to g_∞ in K. The lemma is proved.

Remark. One can easily prove the existence of hermitian orbifold metrics on a line bundle as above by unit partition. The following proposition can be easily proved by modifying the proof of ([Ho] p. 92, Theorem 4.4.1) with the use of the Bochner-Kodaira Laplacian formula (see e.g. [KM]).

Proposition 5.1. *Suppose that (X, g) be a complete Kähler orbifold of complex dimension n, L be a line bundle on X with the hermitian orbifold metric h, and ψ be a function on X, which can be approximated by a decreasing sequence of smooth function $\{\psi_l\}_{1 \leqq l < +\infty}$. If, for any tangent vector v of type $(1, 0)$ at any point of X and for each l,*

$$\left\langle \partial\bar{\partial}\psi_l + \frac{2\pi}{\sqrt{-1}}(\mathrm{Ric}(h) + \mathrm{Ric}(g)), v \wedge \bar{v} \right\rangle_g \geqq C\|v\|^2_g \tag{5.3}$$

where C is a constant independent of l and $\langle , \rangle_g$ is the inner product induced by g. Then for any C^∞ L-valued $(0, 1)$-form w on X with $\bar{\partial}w = 0$ and $\int_X \|w\|^2 e^{-\psi} dV_g$ finite, there exists a C^∞ L-valued function u on X such that $\bar{\partial}u = w$ and

$$\int_X \|u\|^2 e^{-\psi} dV_g \leqq \frac{1}{C}\int_X \|w\|^2 e^{-\psi} dV_g \tag{5.4}$$

where $\|\cdot\|$ is the norm induced by h and g.

Lemma 5.2. *Any section S in $H^0(M_\infty, K_{M_\infty}^{-m})$ is the limit of some sequence $\{S^i\}$ with S^i in $H^0(M_i, K_{M_i}^{-m})$. In particular, it implies that the dimension of $H^0(M_\infty, K_{M_\infty}^{-m})$ is same as that of $H^0(M_i, K_{M_i}^{-m})$, that is, plurianticanonical dimensions are invariant under the degeneration of Kähler-Einstein manifolds with positive scalar curvature.*

Proof. We may assume that $\int_{M_\infty} \|S\|^2_{g_\infty}(x)dV_{g_\infty} = 1$. Let $\{r_i\}$ be a sequence of positive numbers with $\lim_{i\to\infty} r_i = 0$ such that for each i, there is a diffeomorphism ϕ_i from $M_i \setminus \bigcup_{\beta=1}^N B_{r_i}(x_{i\beta}, g_i)$ into $M_\infty \setminus \mathrm{Sing}(M_\infty)$ as given in Lemma 3.3, where

N is defined in Lemma 3.2 and $(x_{i\beta})$ are defined in (3.14). Then ϕ_i satisfy (1) $\lim_{i\to\infty}(\mathrm{Im}(\phi_i))$ is just $M_\infty \setminus \mathrm{Sing}(M_\infty)$; (2) $(\phi_i^{-1})^* g_i$ uniformly converge to g_∞ on any compact subset of $M_\infty \setminus \mathrm{Sing}(M_\infty)$ in C^∞-topology; (3) $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to J_∞, where J_i, J_∞ are almost complex structures on M_i, M_∞, respectively. Define a cut-off function η: $R^1 \to R^1_+$ satisfying: $\eta(t) = 0$ for $t \leqq 1$; $\eta(t) = 1$ for $t \geqq 2$ and $|\eta'| \leqq 1$. Also let π_i be the natural projection from the bundle $\otimes^m(\Lambda^2(TM_i \otimes \overline{TM_i})$ onto $K_{M_i}^{-m} = \otimes^m(\Lambda^2 TM_i)$. For each i, we have a smooth section $v_i = \eta\left(\frac{\rho_i(x)}{2r_i}\right) \cdot \pi_i((\phi_i^{-1})_* S)$ of $K_{M_i}^{-m}$ on M_i, where $\rho_i(x)$ is a Lipschitz function defined by $\rho_i(x) = \min_{1 \leqq \beta \leqq N}\{\mathrm{dist}_{g_i}(x, x_{i\beta})\}$. Then by the facts (2) and (3) above, there is a decreasing function $\varepsilon_3(r)$ on r with $\lim_{r\to 0} \varepsilon_3(r) = 0$ such that

$$\sup\{\|\bar\partial_i \pi_i((\phi_i^{-1})_* S)\|_{g_i}(x) | x \in M_i \setminus \bigcup_{\beta=1}^N B_{2r_i}(x_{i\beta}, g_i)\} \leqq \varepsilon_3(r_i) \tag{5.5}$$

$$\left| \int_{M_i} \|v_i\|^2_{g_i}(x) dV_{g_i} - 1 \right| \leqq \varepsilon_3(r_i) \tag{5.6}$$

where $\bar\partial_i$ is the corresponding $\bar\partial$-operator on M_i.

By (5.5), we have

$$\begin{aligned}
&\int_{M_i} \|\bar\partial_i v_i\|^2_{g_i}(x) dV_{g_i} \\
&\leqq \varepsilon_3(r_i)\mathrm{Vol}_{g_i}(M_i) + \sum_{\beta=1}^N \int_{B_{4r_i}(x_{i\beta}, g_i)} \left\| \bar\partial_i\left(\eta\left(\frac{\rho_i}{2r_i}\right)\right) \cdot \pi_i((\phi_i^{-1})_* S)\right\|^2_{g_i}(x) dV_{g_i} \\
&\leqq \sum_{\beta=1}^N \frac{1}{4r_i^2} \mathrm{Vol}(B_{4r_i}(x_{i\beta}, g_i)) \cdot \sup\{\|(\phi_i^{-1})_* S\|^2_{g_i}(x) | x \in M_i \\
&\setminus \bigcup_{\beta=1}^N B_{2r_i}(x_{i\beta}, g_i)\} + \varepsilon_3(r_i)\mathrm{Vol}_{g_i}(M_i)
\end{aligned} \tag{5.7}$$

As in the proof of Lemma 5.1, one may bound $\sup_{M_\infty}(\|S\|^2_{g_\infty}(x))$ by the constant $C(m)$ in (5.2). Thus by (5.7), Volume Comparison Theorem and the convergence of $(\phi_i^{-1})^* g_i$ in the above fact (2), there is a constant C independent of i such that

$$\int_{M_i} \|\bar\partial_i v_i\|^2_{g_i}(x) dV_{g_i} \leqq C(r_i^2 + \varepsilon_3(r_i)) . \tag{5.8}$$

Now applying Proposition 5.1 i.e., L^2-estimate of $\bar\partial$-operators, we have a C^∞-smooth $K_{M_i}^{-m}$-valued function u_i such that

$$\begin{cases} \bar\partial u_i = \bar\partial v_i, & (5.9) \\ \displaystyle\int_{M_i} \|u_i\|^2_{g_i}(x) dV_{g_i} \leqq \frac{1}{m+1} \int_{M_i} \|\bar\partial_i v_i\|^2_{g_i}(x) dV_{g_i} \leqq \frac{C}{m+1}(r_i^2 + \varepsilon(r_i)) & (5.10) \end{cases}$$

By (5.9), the norm function $\|u_i\|^2_{g_i}$ for each i satisfies an elliptic equation

$$\Delta_i(\|u_i\|^2_{g_i}(x)) = \|D_i u_i\|^2_{g_i}(x) - 2m\|u_i\|^2_{g_i}(x) + 2\mathrm{Re}(h_i^m(u_i, \bar\partial_i^* \bar\partial_i v_i))(x) \tag{5.11}$$

where $\bar{\partial}_i^*$ is the adjoint operator of $\bar{\partial}_i$ on $K_{M_i}^{-m}$-valued function with respect to g_i. As in (5.5), we also have

$$\sup\{\|\bar{\partial}_i^* \bar{\partial}_i v_i\|_{g_i}^2(x) | x \in M_i \setminus B_{4r_i}(x_{i\beta}, g_i)\} \to 0 \quad \text{as} \quad i \to \infty \tag{5.12}$$

Combining (5.9) and (5.10), we see that u_i converge uniformly to zero in the sense of the remark after Lemma 5.1 as i goes to infinity. Put

$$S^i(x) = \frac{(v_i(x) - u_i(x))}{\left(\int_{M_i} \|v_i - u_i\|_{g_i}^2(x) dV_{g_i}\right)^{\frac{1}{2}}} \tag{5.13}$$

Then $\{S^i\}$ is the required sequence.

Lemma 5.3. *Let $\{(M_i, g_i)\}$ and (M_∞, g_∞) be given in Proposition 4.2. For each integer $m > 0$, we have othonormal bases $\{S_{m\beta}^i\}_{0 \leq \beta \leq N_m}$ (resp. $\{S_{m\beta}^\infty\}$) of $H^0(M_i, K_{M_i}^{-m})$ (resp. $H^0(M_\infty, K_{M_\infty}^{-m})$). Then*

$$\lim_{i \to \infty} \left(\inf_{M_i} \left\{ \sum_{\beta=0}^{N_m} \|S_{m\beta}^i\|_{g_i}^2(x) \right\} \right) \geqq \inf_{M_\infty} \left\{ \sum_{\beta=0}^{N_m} \|S_{m\beta}^\infty\|_{g_\infty}^2(x) \right\}. \tag{5.14}$$

Proof. By direct computations, we have

$$\Delta_i(\|D_i S_{m\beta}^i\|_{g_i}^2)(x) = \|D_i D_i S_{m\beta}^i\|_{g_i}^2(x) - (4m-1)\|D_i S_{m\beta}^i\|_{g_i}^2(x) \tag{5.15}$$

where Δ_i (resp. D_i) is laplacian (resp. covariant derivative) with respect to g_i. Then by (5.1), Lemma 2.1 and standard Moser's iteration, there is a constant $C'(m)$ depending only on m such that

$$\sup\{\|D_i S_{m\beta}^i\|_{g_i}^2(x) | 0 \leqq \beta \leqq N_m, x \in M_i\} \leqq C'(m). \tag{5.16}$$

Combining it with (5.2), we conclude that the first derivatives of $\sum_{\beta=0}^{N_m} \|S_{m\beta}^i\|_{g_i}^2(x)$ are uniformly bounded independent of i. Then (5.14) follows from this and Lemma 5.1, 5.2.

As a corollary of this lemma, we have the following weak partial C^0-estimates of the solution of $(1.1)_t$.

Theorem 5.1. *There are a universal integer $m_0 > 0$ and a universal constant $C > 0$ such that for any Kähler-Einstein surface (M', g') in $\mathfrak{I}_n (5 \leqq n \leqq 8)$, we have*

$$\inf_{M'} \left\{ \sum_{\beta=0}^{N_m} \|S_\beta'\|_{g'}^2 \right\} \geqq C > 0 \tag{5.17}$$

where $N_m + 1$ is the complex dimension of $H^0(M', K_{M'}^{-m_0})$ and $\{S_\beta'\}_{0 \leq \beta \leq N}$ is an orthonormal basis of $H^0(M', K_{M'}^{-m_0})$ with respect to the inner product induced by g'.

Proof. It suffices to prove that for any sequence of Kähler-Einstein surface $\{(M_i, g_i)\}$ converging to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Proposition 4.2, there are $m_0 > 0$ and $C > 0$ such that (5.17) holds for these (M_i, g_i). By Lemma 5.3, it is sufficient to find a large m such that

$$\inf \left\{ \sum_{\gamma=0}^{N_m} \|S_{m\gamma}^\infty\|^2(x) | x \in M_\infty \right\} > 0 \tag{5.18}$$

where $\{S^{\infty}_{m\gamma}\}$, N_m are given as in Lemma 5.3. It is equivalent to that for any point x in M_∞, there is a holomorphic global section S in $H^0(M_\infty, K^{-m}_{M_\infty})$ such that $S(x) \neq 0$. The latter can be achieved by the application of L^2-estimate (Proposition 5.1) as follows. Let $x_{\infty 1}, \ldots, x_{\infty N}$ be the singular points of M_∞. There is a small positive number r independent of β such that for any $x_{\infty\beta}$ in M_∞, the closure of each connected component in $B_r(x_{\infty\beta}, g_\infty)\backslash\{x_{\infty\beta}\}$ is locally uniformized by a neighborhood $\tilde{U}_{\beta j}(1 \leqq j \leqq l_\beta)$ of the origin o in C^2 with finite uniformization group Γ_β. Let $\pi_{\beta j}: \tilde{U}_{\beta j} \to B_r(x_{\infty\beta}, g_\infty)$ be the natural projection with $\pi_{\beta j}(o) = x_{\infty\beta}$, and $q = \prod_{1 \leqq \beta \leqq N}(\prod_{1 \leqq j \leqq R_\beta} q_{\beta j})$, where $q_{\beta j}$ is the order of the finite group $\Gamma_{\beta j}$. Let $m = pq$. We will choose p later. We may take r to be sufficiently small such that the function $\rho\beta = \mathrm{dist}(\cdot, x_{\infty i})^2$ is smooth on $B_r(x_{\infty\beta}, g_\infty)\backslash\{x_{\infty\beta}\}$ for any β. Now fix a $x_{\infty\beta}$ and $\tilde{U}_{\beta j}$.

Let (z_1, z_2) be a coordinate system on $\tilde{U}_{\beta j}$, define an n-anticanonical section v by

$$v(y) = \sum_{\sigma \in \Gamma_{\beta j}} \sigma^*\left(\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right)^q\right)(y), \quad y \in \tilde{U}_{\beta j}.$$

By the definition of q, we have $v(o) \neq 0$. Let $\eta: R^1 \to R^1_+$ be a cut-off function such that $\eta(t) = 1$ for $t \leqq 1$; $\eta(t) = 0$ for $t \geqq 2$. $|\eta'(t)| \leqq 1$. Then $w = \eta\left(\frac{4\rho\beta}{r^2}\right)(\pi_{\beta j})_*(v^p)$ is a C^∞-global section of the line bundle $K^{-m}_{M_\infty}$. Choose a large p depending only on r such that for tangent vector v of type $(1, 0)$,

$$\left\langle \partial\bar{\partial}\left(8\eta\left(\frac{4\rho\beta}{r^2}\right)\log\left(\frac{\rho\beta}{r^2}\right)\right) + \frac{2\pi p}{\sqrt{-1}}\omega_{g_\infty}, v \wedge \bar{v} \right\rangle_{g_\infty} \geqq \|v\|^2_{g_\infty}. \tag{5.19}$$

Applying Proposition 5.1, we obtain a C^∞ smooth $K^{-m}_{M_\infty}$-valued function u satisfying $\bar{\partial}u = \bar{\partial}w$ and

$$\int_{M_\infty} \|u\|^2_{g_\infty} e^{-8\eta\log\left(\frac{\rho\beta}{r^2}\right)} dV_{g_\infty} \leqq \int \|\bar{\partial}w\|^2_{g_\infty} e^{-8\eta\log\left(\frac{\rho\beta}{r^2}\right)} dV_{g_\infty} < +\infty.$$

It follows that the pull-back $\pi^*_{\beta j}u$ of u vanishes up to order 3 at the origin in $\tilde{U}_{\beta j} \subset C^2$. Put

$$S_{\beta j} = \frac{w - u}{\left(\int_{M_\infty} \|w - u\|^2_{g_\infty} dV_{g_\infty}\right)^{\frac{1}{2}}} \tag{5.20}$$

then $S_{\beta j} \in H^0(M_\infty, K^m_{M_\infty})$ and $\inf_{\tilde{U}_{\beta j}}\{\pi^*_{\beta j}\|S_{\beta j}\|_{g_\infty}(x)\} > 0$. By the same arguments as in the proof of Lemma 5.3, one can bound the first derivatives of these $S_{\beta j}$ by a uniform constant. So if r is taken sufficiently small, we have

$$\begin{aligned}
&\inf\left\{\sum_{\gamma=0}^{N_m} \|S^{\infty}_{m\gamma}\|^2_{g_\infty}(x) | x \in B_r(x_{\infty\beta}, g_\infty),\ 1 \leqq \beta \leqq N_m\right\} \\
&\geqq \inf\{\|S_{\beta j}\|^2_{g_\infty}(x) | x \in \pi_{\beta j}(\tilde{U}_{\beta j}),\ 1 \leqq \beta \leqq N_m,\ 1 \leqq j \leqq l_\beta\} \\
&> 0.
\end{aligned}$$

For any point x in $M_\infty \setminus \bigcup_{\beta=1}^{N} B_r(x_{\infty\beta}, g_\infty)$, define $\rho_x = \mathrm{dist}(\cdot, x)^2$. As above, applying Proposition 5.1 to $K_{M_\infty}^{-m}$-valued $\bar\partial$-equation with the weight function $8\eta\left(\frac{4\rho_x}{r^2}\right)\log\left(\frac{\rho_x}{r^2}\right)$, one can easily construct a holomorphic section S_x in $H^0(M_\infty, K_{M_\infty}^{-m})$ such that $S_x(x) \neq 0$. Thus the inequality (5.18) is proved. So is Theorem 5.1.

Proposition 5.2. *The Kähler-Einstein orbifold (M_∞, g_∞) in Proposition* 4.2 *is locally irreducible, that is, for any $r > 0$ and any point x in M_∞, the punctured ball $B_r(x, g_\infty)\setminus\{x\}$ is connected. In particular, the orbifold M_∞ is irreducible.*

Proof. It suffices to check the local irreducibility at a singular point x, say $x = x_{\infty 1}$, in M_∞. Suppose that M_∞ is not locally irreducible at $x = x_{\infty 1}$, then we have open subsets $\tilde U_{11}, \ldots, \tilde U_{1l_1}$ ($l_1 \geqq 2$) uniforming the closures of the connected components in $B_r(x, g_\infty)\setminus\{x\}$. In the above proof of Theorem 5.1, we construct a S in $H^0(M_\infty, K_{M_\infty}^m)$ for $m = m_0$ such that it can be decomposed into $v + u$ in $B_{\frac{r}{2}}(x, g_\infty)\setminus\{x\}$. Both v, u are holomorphic in $B_{\frac{r}{2}}(x_\infty, g_\infty)\setminus\{x_{\infty 1}\}$ and satisfy (1) $v = 0$ on $\pi_{1j}(\tilde U_{1j})(j \geqq 2)$, and $\inf\{\|v(y)\| \mid y \in \pi_{11}(\tilde U_{11})\} \geqq c' > 0$; (2) $\|u\|_{g_\infty}(y) \leqq C\,\mathrm{dist}(y, x_1)^4$ for a constant C. In particular, it implies that for any sufficient small $r' > 0$, we have a uniform lower bound $\frac{c'}{2}$ of the oscillation,

$$\omega_{r'}(S, g_\infty) = \sup\{|\|S\|_{g_\infty}(y) - \|S\|_{g_\infty}(z)| \mid y, z \in \partial B_{r'}(x_1, g_\infty)\} \geqq \frac{c'}{2}$$

By Lemma 5.2, there is a sequence $\{S^i\}$ with S^i in $H^0(M_i, K_{M_i}^{-m})$ such that S^i converge to S as M_i converge to M_∞. Then for any fixed small r',

$$\lim_{i\to\infty} \omega_{r'}(S^i, g_i) = \omega_{r'}(S, g_\infty) \geqq \frac{c'}{2}. \tag{5.21}$$

On the other hand, in the proof of Lemma 5.3, we proved that there is a constant $C(m)$ depending only on m, such that

$$\sup_{M_i}(\|\nabla_i(\|S^i\|_{g_i})\|_{g_i}) \leqq C(m).$$

It follows that $\omega_{r'}(S', g_i) \leqq 2C(m)r'$. It certainly contradicts to (5.21) when r' is sufficiently small. So M_∞ is locally irreducible at $x_{\infty 1}$, similarly at $x_{\infty j}$ ($2 \leqq j \leqq l_1$). The proposition is proved.

Remark. Let $x_{i\beta} \in M_i (1 \leqq \beta \leqq N)$ be given in (3.14), and $\tilde B_r(x_{i\beta}, g_i)$ be the universal covering of the ball $B_r(x_{i\beta}, g_i)$ in M_i. Then, as Proposition 4.2, for any fixed β, these $\tilde B_r(x_{i\beta}, g_i)$ converge to an open Kähler-Einstein orbifold $\tilde B_\infty(r)$. The above arguments also prove that this $\tilde B_\infty(r)$ is locally irreducible, in particular irreducible. As a consequence, it implies that $\partial\tilde B_r(x_{i\beta}, g_i)$ is connected if r is small and the fundamental group $\pi_1(B_r(x_{i\beta}, g_i))$ is a quotient group of $\pi_1(\partial B_r(x_{i\beta}, g_i))$ by some normal subgroups.

Now we come to study the singularities $\{x_{\infty\beta}\}_{1\leqq\beta\leqq N}$ of the Kähler-Einstein orbifold (M_∞, g_∞) in Proposition 4.2, where N is given as in Lemma 3.2. Precisely,

we want to make reduction of those local uniformization groups Γ_β for $\{x_{\infty\beta}\}_{1\leq\beta\leq N}$.

Choose a small $\tilde{r}>0$, such that for any β between 1 and N, the ball $B_{\tilde{r}}(x_{\infty\beta}, g_\infty)$ is geodesically convex and is locally uniformized by an open subset $\tilde{U}_\beta$ in C^2 with the local uniformization group Γ_β. We identify Γ_β with the induced action of Γ_β on $T_o\tilde{U}_\beta$, where o is the preimage of $x_{\infty\beta}$ under the natural projection $\pi_\beta\colon \tilde{U}_\beta \to B_{\tilde{r}}(x_{\infty\beta}, g_\infty)$. Then Γ_β can be considered as a finite subgroup in $U(2)$. Let $\{r_i\}$ be a decreasing sequence with $\lim_{i\to\infty} r_i = 0$, they by Proposition 4.2, there are diffeomorphisms ϕ_i from $M_i/\bigcup_{\beta=1}^N B_{2r_i}(x_{\infty\beta}, g_\infty)$ and $(\phi_i^{-1})^* g_i$ converge pointwisely to g_∞ in C^∞-topology. We may assume that all $r_i < \frac{\tilde{r}}{2}$.

Fix a singular point $x = x_{\infty\beta}$, say $\beta = 1$ for simplicity, we define $S_r(i)$ to be $\phi_i^{-1}(\partial B_r(x, g_\infty))$ for $2r_i \leqq r \leqq \tilde{r}$. Then each $S_r(i)$ is isomorphic to a generalized lens space S^3/Γ with $\Gamma = \Gamma_1$ and covers a domain $B_r(i)$ such that $B_r(i)$ converge to $B_r(x, g_\infty)$ as i goes to infinity in the sense of Proposition 4.2. By taking $\tilde{r}$ smaller, we may assume that each $B_r(i)$ is geodesically convex. As mentioned in the above remark, by the same proof as that for Theorem 5.1, one can prove that for any $r>0$ and i, the fundamental group $\pi_1(B_r(i))$ is the quotient group of Γ_1 by its normal subgroup. In particular, the group $\pi_1(B_r(i))$ is a finite group with its order uniformly bounded. We may assume that the orders of these $\pi_1(B_r(i))$ are all same.

Lemma 5.4. *Let $pr_i\colon \tilde{B}_{\tilde{r}}(i) \to B_{\tilde{r}}(i)$ be the universal covering and $\tilde{g}_i = pr_i^* g_i$. Then $(\tilde{B}_{\tilde{r}}(i), \tilde{g}_i)$ converge to an open Kähler-Einstein orbifold $(\tilde{B}_{\tilde{r}}(\infty), \tilde{g})$ in the sense of Proposition 4.2. Moreover, there is a natural projection $p_{r_\infty}\colon \tilde{B}_{\tilde{r}}(g_\infty) \to B_{\tilde{r}}(x, \infty)$ of order $\# \pi_1(B_{\tilde{r}}(i))$ and $\tilde{g}_\infty = p_{r_\infty}^* g_\infty$. and $\tilde{B}_{\tilde{r}}(\infty)$ has a rational double point as the only singular point.*

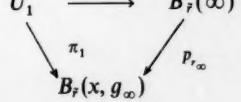
Proof. The convergence of $\{(\tilde{B}_{\tilde{r}}(i), \tilde{g}_i)\}$ follows from Proposition 4.2 and the definitions of $\tilde{B}_{\tilde{r}}(i)$ and $\tilde{B}_{\tilde{r}}(i)$. It is clear that $\tilde{B}_{\tilde{r}}(\infty)$ has only one singular point. So it suffices to prove that this singular point is a rational double point. By Theorem 5.1, we have holomorphic sections S^i in $H^0(M_i, K_{M_i}^{-m_0})$ such that when $r>0$ is sufficiently small, there is a positive number $c>0$,

$$0 < c \leqq \|S^i\|_{g_i}^2(x) \leqq 1 \quad \text{for } x \in B_r(i) \tag{5.21}$$

Then each $pr_i^* S^i$ is a holomorphic m_0-anticanonical section on $\tilde{B}_{\tilde{r}}(i)$. Since the preimage $(p_{r_i})^{-1}(B_r(i))$ is simply-connected, the m_0-root of $p_{r_i}^* S^i$ exists as a holomorphic anticanonical section on $((pr_i)^{-1}(B_r(i))$, denoted by $\tilde{S}^i$. Then

$$0 < c^{\frac{1}{m_0}} \leqq \|\tilde{S}^i\|_{g_i}^2(x) \leqq 1 \quad \text{for } x \in (p_{r_i})^{-1}(B_r(i)) \tag{5.22}$$

By (5.22), we may assume that $\tilde{S}^i$ converge to a nonvanishing holomorphic anticanonical section $\tilde{S}$ in $(p_{r_\infty})^{-1}(B_r(x, g_\infty))$. The local uniformization $\tilde{U}_1$ of x in M_∞ is also the one for $(p_{r_\infty})^{-1}(x)$ in $\tilde{B}_{\tilde{r}}(\infty)$, and we have the following commutative diagram,



Let Γ' be the local uniformization group of $(p_{r_\infty})^{-1}(x)$ in $\tilde{B}_{\bar{r}}(\infty)$. Then $\tilde{\pi}_1^* \tilde{S}$ is a Γ'-invariant holomorphic anticanonical section on $\pi_1^{-1}(B_r(x, g_\infty)) \subset U_1$. Since $\tilde{\pi}_1^* \tilde{S} \neq 0$, the induced action of Γ' on $\Lambda^2(T_o \tilde{U}_1)$ is trivial. This means that Γ' is a finite group in $SU(2)$ if we identify Γ' with its induced group on $T_o \tilde{U}_1 \cong C^2$. So the singular point $p_{r_\infty}^{-1}(x)$ is a rational double point. The lemma is proved.

Lemma 5.5. *The induced group of Γ on $T_o \tilde{U}_1 \cong C^2$ is either a finite subgroup in $SU(2)$ or one of the following cyclic group $Z_{l,p,q}$ defined as follows, where p, q are coprime, let (z_1, z_2) be the euclidean coordinates in C^2, define*

$$\sigma_{l,p,q}: C^2 \to C^2,$$

$$\sigma_{l,p,q}((z_1, z_2)) = \left(e^{\frac{2\pi\sqrt{-1}}{lp^2}} z_1, e^{-\frac{2\pi\sqrt{-1}}{lp^2} + \frac{2\pi q\sqrt{-1}}{p}} z_2 \right)$$

then $Z_{l,p,q}$ is generated by $\sigma_{l,p,q}$.

Proof. Let ϕ_i be the diffeomorphisms given before Lemma 5.4. There is a decreasing sequence $\{\varepsilon_i\}_{i \geq 1}$ with $\lim_{i \to \infty} \varepsilon_i = 0$ such that

$$\sup_{\substack{1 \leq k \leq 5 \\ y \in M_\infty \setminus \bigcup_{\beta=1}^N B_{r_i}(x_{\infty\beta}, g_\infty)}} \{ \| (\phi_i^{-1})^* g_i - g_\infty \|_{g_\infty}(y), \| D^k (\phi_i^{-1})^* (g_i) \|_{g_\infty}(y) \} \leqq \varepsilon_i . \tag{5.23}$$

where D is the covariant derivative with respect to g_∞. Put $\mu_i = \dfrac{1}{\sqrt{r_i}}$. Then

$$\sup_{\substack{1 \leq k \leq N \\ y \in B_{\bar{r}}(x, g_\infty) \setminus B_{r_i}(x, g_\infty)}} \{ \| (\phi_i^{-1})^* (\mu_i g_i) - \mu_i g_\infty \|_{\mu_i g_\infty}(y), \| D_\mu^k (\phi_i^{-1})^* (\mu_i g_i) \|_{\mu_i g_\infty}(y) \} \leqq \varepsilon_i \tag{5.24}$$

where D_μ is the covariant derivative with respect to g_∞. Since the curvature tensor $R(g_\infty)$ of g_∞ is uniformly bounded on M_∞, the dilated manifolds $(B_{\bar{r}}(x, g_\infty), \mu_i g_\infty)$ converge to the flat cone C^2/Γ_1 with complete flat metric g_F. So it follows from (5.23) that $(B_{\bar{r}}(i), \mu_i g_i)$ converge to $(C^2/\Gamma_1, g_F)$. Similarly, the Kähler-Einstein manifolds $(B_{\bar{r}}(i), \mu_i p r_i^* g_i)$ converge to the cone $(C^2/\Gamma', g_F)$ with $\Gamma' \subset SU(2)$. The fundamental groups $\pi_1(B_{\bar{r}}(i))$ can be regarded as the finite isometry groups on $\tilde{B}_{\bar{r}}(i)$. Then the actions of these $\pi_1(B_{\bar{r}}(i))$ converge to the linear action of $\bar{\Gamma} = \Gamma_1/\Gamma'$ on C^2/Γ'. Note that Γ' is a normal subgroup of Γ.

It is classical result by Klein that C^2/Γ' is one of the following normal hypersrfaces in C^3 with euclidean coordinates (w_1, w_2, w_3),

$$\begin{aligned}
A_{n-1} (n \geqq 2)&: w_1 w_3 + w_2^n = 0 \text{ or } w_3^2 + w_1^2 + w_1^2 = 0 \\
D_n (n \geqq 4)&: w_3^2 + w_2 (w_1^2 + w_2^{n-1}) = 0 \\
E_6&: w_3^2 + w_1^3 + w_2^4 = 0 \\
E_7&: w_3^2 + w_1 (w_1^2 + w_2^3) = 0 \\
E_8&: w_3^2 + w_1^3 + w_2^5 = 0
\end{aligned} \tag{5.25}$$

Claim 1. The finite group $\tilde{\Gamma}$ has a faithful representation in $SL(3, C)$ such that the action of $\tilde{\Gamma}$ on C^2/Γ' coincides with its restriction to the hypersurface corresponding to C^2/Γ'.

Although this claim should be the special case of a more general theorem, we give an elementary proof here.

As before, let (z_1, z_2) be the euclidean coordinates of C^2. Let $C_{\Gamma'}[z_1, z_2]$ be the algebra of all Γ'-invariant polynomials in $C[z_1, z_2]$. Then the result of Klein actually says that $C_{\Gamma'}[z_1, z_2]$ is generated by three homogeneous polynomials, i.e. induced ones by restriction to C^2/Γ' of coordinates w_i on C^3. We still denote them by w_1, w_2, w_3. Since Γ' is a normal subgroup of Γ, any $\sigma \in \Gamma$ preserves the subalgebra $C_{\Gamma'}[z_1, z_2]$ of the polynomial algebra $C[z_1, z_2]$. Thus Γ has a holomorphic action on C^3 with its restriction to C^2/Γ equal to the original action of Γ. We need to prove that this induced action σ^* on C^3 is linear. By the explicit forms of the defining polynomials in (5.25), one can easily prove that as polynomials of (z_1, z_2),

$$\deg(w_2) \leqq \deg(w_1) \leqq \deg(w_3)\deg(w_3) < 2\deg(w_1) \tag{5.26}$$

If $\deg(w_3) = \deg(w_2)$, then $\sigma^* w_i$ is a linear combination of w_1, w_2 and w_3 for any $\sigma \in \Gamma$. Thus we may assume that $\deg(w_3) > \deg(w_2)$. If $\deg(w_1) = \deg(w_2)$, then for any $\sigma \in \Gamma$, $\sigma^* w_i$ depends linearly on w_1, w_2 for any $i = 1, 2$. On the other hand, $\sigma^* w_3$ can not contain w_1, w_2 since σ^* preserves the defining equations in (5.25), so σ^* is linear. If $\deg(w_1) < \deg(w_3)$, then by the second inequality in (5.26), the polynomial $\sigma^*(w_3)$ does not depend on w_1 and $\sigma^*(w_i)$ does not depend on w_3 for $i = 1, 2$. Thus by the fact that σ^* preserves one of the equations in (5.25), one can easily see that $\sigma^*(w_j) = \lambda_j w_j$ for some constants λ_j $(j = 1, 2, 3)$, in particular, the action of Γ is linear. The final case is that $\deg(w_1) = \deg(w_3)$, $\deg(w_2) < \deg(w_3)$. In this case, the group Γ' is of type $A_n (n \geqq 2)$, i.e., the hypersurface C^2/Γ' is defined by A_n-type polynomials in (5.25). Then $w_1 = z_1^n$, $w_2 = z_1 z_2$, $w_3 = z_2^n$. By the Γ'-invariance of $\sigma^* w_j$, one can easily prove that σ^* is linear. The faithfulness can be easily proved. The claim is proved.

Note that Γ' is the subgroup of Γ consisting of all elements with determinant one, so $\tilde{\Gamma} = \Gamma/\Gamma'$ is a cyclic group. Let σ be one of its generators. By the above proof of Claim1, we actually proved that the induced action σ^* on C^3 is diagonal except that Γ' is of type A_1. But if Γ' is of type A_1, all w_i have the same degree on (z_1, z_2), so by a linear transformation of (w_1, w_2, w_3), we may also assume that the action of Γ is a diagonal and the defining equation of C^2/Γ' is of form given in (5.25).

Write $\sigma = \operatorname{diag}\left(e\left(\frac{p_1}{p}\right), e\left(\frac{p_2}{p}\right), e\left(\frac{p_3}{p}\right)\right)$, where $e\left(\frac{p_i}{p}\right) = e^{\frac{2\pi\sqrt{-1}}{p}}$. Consider the representation of Γ in $S^1 \in C^*$ defined as follows:

$$\rho : \tau \in \Gamma \to \det(\tau) \in S^1 \subset C^*$$

where the determinant is taken with respect to (z_1, z_2). Then $\ker(\rho) = \Gamma'$ and the induced representation of ρ on Γ/Γ' is faithful. We still denote it by ρ. We may assume that $\rho(\sigma)\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right) = e\left(-\frac{1}{p}\right)\left(\frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_2}\right)$. So $u_j = w_j\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right)^{p_j}$

$(j = 1, 2, 3)$ are Γ-invariant holomorphic local sections of $K_{C^2/\Gamma'}^{-p_j}$ on C^2/Γ'.

Let ϕ_i be the diffeomorphisms as those at the beginning of this proof. Put $\rho_i(y) = \rho_\infty \circ \phi_i(y)$, where ρ_∞ is the square of the standard distance function on C^2. Then for i sufficiently large, the function ρ_i is plurisubharmonic on $B_{4\sqrt{r_i}}(i)/B_{4r_i}(i)$, and these ρ_i converge to ρ_∞.

As in the proof of Lemma 5.3, we let $\bar\partial_i$ be the $\bar\partial$-operator associated to M_i and π_i be the projection from $\otimes^{p_i}(\Lambda^2(TM_i \oplus \overline{TM_i}))$ onto $K_{M_i}^{-p_i}$. Define

$$v_{ij} = \bar\partial_i\left(\eta\left(\frac{\rho_i}{2r_i}\right)\pi_i((\phi_i^{-1})_*(u_j))\right)$$

where η is a cut-off function on R^1 with $\eta(t) = 0$ for $t \leqq 1$ and $\eta(t) = 1$ for $t \geqq 1$, $|\eta'(t)| \leqq 1$. Let ψ be an increasing function on $(-\infty, 4)$ such that $\psi(t) = 0$ for $t \leqq 1$ and $\psi(t)$ goes to $+\infty$ very fast as $t \to 4$. As in the proof of Lemma 5.3, we apply L^2-estimate of $\bar\partial$-operators with weight function $\psi \circ \rho_i$ (Proposition 5.1) to the equations $\bar\partial_i u = v_{ij}$ on $\phi_i^{-1}(B_{2\sqrt{r_i}}(x_{\infty 1}, g_\infty)) \cup B_{4r_i}(i)$ and obtain local holomorphic anticanonical sections u_{ij} on $(B_{\sqrt{r_i}}(i), \mu_i g_i)$ $(j = 1, 2, 3)$ such that u_{ij} converge to u_j as $(B_{\sqrt{r_i}}(i), \mu_i g_i)$ converge to $(\{\rho_\infty < 1\}, g_F) \subset (C^2/\Gamma, g_F)$. Recall that $pr_i : \tilde B_{\sqrt{r_i}}(i) \to B_{\sqrt{r_i}}(i)$ are universal coverings and there are nonvanishing holomorphic sections $\tilde S^i$ of $K_{\tilde B_{\sqrt{r_i}}(i)}^{-1}$ (cf. the proof of Lemma 5.4). Without losing generality, we may assume that $\tilde S^i$ converge to $\dfrac{\partial}{\partial z_1} \wedge \dfrac{\partial}{\partial z_2}$ on C^2/Γ' as i goes to infinity. Thus the holomorphic functions $w_{ij} = pr_i^*(u_{ij})/(S^i)^{p_j}$ on $(\tilde B_{\sqrt{r_i}}(i), \mu_i g_i)$ converge to w_j on C^2/Γ'.

Claim 2. For any sufficiently large i, the functions $\{w_{ij}\}_{1 \leqq j \leqq 3}$ give an embedding Ψ_i of $\tilde B_{\sqrt{r_i}}(i)$ into C^3 satisfying:

(1) A generator σ in $\pi_1(B_F(i))$ can be taken such that it acts on C^3 by a diagonal matrix $\sigma^*(w_{ij}) = e\left(\dfrac{p_j}{p}\right)w_{ij}$. Note that $\pi_1(B_F(i))$ is a cyclic group.

(2) $\Psi_i(\tilde B_{\sqrt{r_i}}(i))$ converge to the open subset $\{\rho_\infty < 1\}$ in C^2/Γ in Hausdorff topology.

The map Ψ_i is defined by assigning y in $\tilde B_{\sqrt{r_i}}(i)$ to $(w_{i1}(y), w_{i2}(y), w_{i3}(y))$ in C^3. Since $\{(\phi_i^{-1})^* w_{ij}\}_{1 \leqq j \leqq 3}$ converge uniformly to $\{w_j\}_{1 \leqq j \leqq 3}$ on $\{\varepsilon \leqq \rho_\infty < 1\}$ for any $\varepsilon > 0$, the map Ψ_i is an embedding on $\tilde B_{\sqrt{r_i}}(i)/\tilde B_{\varepsilon\sqrt{r_i}}(i)$ for the sufficiently large i. Because $K_{M_i}^{-1}$ is ample and there is a section in $H^0(M_i, K_{M_i}^{-m})$ vanishing nowhere in $\tilde B_{\sqrt{r_i}}(i)$ for a large m (Theorem 5.1), there is no complete holomorphic curve in $\tilde B_{\sqrt{r_i}}(i)$. It follows that Ψ_i fails to be injective at most at finitely many points in $\tilde B_{\sqrt{r_i}}(i)$. If Ψ_i is not injective, then there are two points y_1, y_2 in $\tilde B_{\sqrt{r_i}}(i)$ such that $\Psi_i(y_1) = \Psi_i(y_2)$. Choose a very small $r > 0$, such that the geodesic balls $B_r(y_i, pr_i^*, \dot g_i)$ do not intersect to each other and Ψ_i is an embedding at any point of these balls except y_1, y_2. Let f_i be the local defining function of $\mathrm{Im}(\Psi_i)$ at $\Psi_i(y_1)$. Then f_i is reducible. Let f_i' (resp. f_i'') be the irreducible component of f_i such that $\{f_i' = 0\}$ (resp. $\{f_i'' = 0\}$) corresponds to $\Psi_i(B_r(y_1, pr_i^* g_i))$ (resp. $\Psi_i(B_r(y_2, pr_i^* g_i))$). Then $\{f_i' = 0\}$ (resp. $\{f_i'' = 0\}$) is smooth outside $\Psi_i(y_1)$ (resp. $\Psi_i(y_2)$). Let D be the

curve defined by $f_i' = 0$ and $f_i'' = 0$. Then Ψ_i fails to be injective along $\Psi_i^{-1}(D)$. A contradiction. Therefore, Ψ_i is an embedding from $\tilde{B}_{\sqrt{r_i}}(i)$ into C^3.

In order to prove the statement (1), it suffices to check that for any r in $\pi_1(B_\Gamma(i))$, we have $r^*(w_{ij}) = \lambda_{ij} w_{ij}$ for some real numbers $\lambda_{ij} (1 \leqq j \leqq 3)$. By definition, we can write $w_{ij} = pr_i^*(u_{ij})/(\tilde{S}_i)^{p_j}$. Since each $pr_i^*(u_{ij})$ is Γ-invariant, we have

$$\tau^* w_{ij} = \frac{pr_i^*(u_{ij})}{(\tau^* \tilde{S}^i)^{p_j}} = \left(\frac{\tilde{S}^i}{\tau^* \tilde{S}^i}\right)^{p_j} w_{ij} = \lambda_{ij} w_{ij}$$

where λ_{ij} are holomorphic functions on $\tilde{B}_{\sqrt{r_i}}(i)$. Because τ is an isometric of the metric $pr_i^* g_i$ and $\tilde{S}^i$ is the m_0-root of a Γ-invariant m_0-anticanonical section, the absolute values of λ_{ij} are identically one, so λ_{ij} are constants.

The statement (2) is then trivially true. The claim is proved.

Now we can complete the proof of this lemma. We identify each $\pi_1(B_\Gamma(i))$ with Γ/Γ' such that $\sigma = \operatorname{diag}\left(e\left(\frac{p_1}{p}\right), e\left(\frac{p_2}{p}\right), e\left(\frac{p_3}{p}\right)\right)$ as a linear transformation on C^3. Let f_i, f_∞ be the defining equations of $\Psi_i(\tilde{B}_{\sqrt{r_i}}(i))$ and C^2/Γ in C^3, respectively. Then f_∞ is one of the polynomials in (5.25) and $\lim_{i\to\infty} f_i = f_\infty$. Since σ preserves the hypersurfaces $\{f_i = 0\}$ and $\{f_\infty = 0\}$, we have that $\sigma^* f_i = \lambda f_i$, $\sigma^* f_\infty = \lambda f_\infty$, where λ is a nonzero constant. If $\lambda \neq 1$, then any f_i has no constant term, in particular, the origin of C^3 is in $\{f_i = 0\}$. It follows that the group $\pi_1(B_\Gamma(i)) = \Gamma_1/\Gamma'$ does not act on $\tilde{B}_{\sqrt{r_i}}(i)$ freely. A contradiction. So $\lambda = 1$ and each f_i must have nonzero constant term. It follows that $\{f_i = 0, w_j = 0\}$ is non empty for any $i \geqq 1$ and $1 \leqq j \leqq 3$. We remark that $\pi_1(B_\Gamma(i))$ acts on $\{f_i = 0\}$ freely, so if $kp_{j'} \equiv 0$ for (mod p) for some j', then $kp_{j'} \equiv 0 \pmod{p}$ for all j, where k is an integer. Thus we may assume that $0 < p_j < p$ for all $j = 1, 2, 3$. If Γ' is of type other than A_n, then $2p_3 \equiv 0 \pmod{p}$, so by the above remark, we have $2p_j = p$ for all j. It follows that $\sigma = \operatorname{diag}(-1, -1, -1)$. It is certainly impossible since the latter diagonal matrix does not preserve the equation of type D_n, E_6, E_7, E_8 in (5.25). Therefore, Γ' must be of type A_n.

If the defining polynomial f_∞ is of form $w_3^2 + w_1^2 + w_2^2 = 0$, then the same argument as above shows that $\sigma = \operatorname{diag}(-1, -1, -1)$.

Now we assume that f_∞ is of form $w_1 w_3 + w_2^n = 0$.

Claim 3. The element σ in $\pi_1(B_\Gamma(i))$ is one of $\sigma_{l,n,q}$ described in the statement of this lemma. In particular, the lemma follows from this claim.

By the fact that the action of $\pi_1(B_\Gamma(i))$ is free of fixed point, we may assume that $p_1 = 1$ and p, p_2 are coprime. Then $n = pn'$ by $\sigma^* f_\infty = f_\infty$. Recall that $w_1 = z_1^n$, $w_3 = z_2^n$, $w_2 = z_1 z_2$ for euclidean coordinates (z_1, z_2) in C^2. As an element in $U(2)$, we can write $\sigma = \operatorname{diag}\left(\frac{1+pq_1}{np}, -\frac{1+pq_1}{np} + \frac{p_2}{p}\right)$. Let m be the largest common factor of $1 + pq_1$ and np, then m, p are coprime. Write $1 + pq_1 = ml_1$, $np = n'p^2 = mlp^2$, where l_1 is coprime to both l and p. Let m_1, m_2 be such that $m_1 l_1 + m_2 lp^2 = 1$ and m_1 is coprime to l, p. So $\sigma^{m_1} = \operatorname{diag}\left(\frac{1}{lp^2}, -\frac{1}{lp^2} + \frac{m_1 p_2}{p}\right)$. Thus we may assume that $\sigma = \sigma_{l,p,q}$. The claim is proved, and so is the lemma.

We summarize the above discussion in the following

Theorem 5.2. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein surfaces in $\mathfrak{I}_n$ with* $\mathrm{Ric}(g_i) = \omega_{g_i}$. *Then by taking a subsequence if necessary, we may have that $(M_i g_i)$ converge to an irreducible Kähler-Einstein orbifold (M_∞, g_∞) (in the sense of Proposition 4.2) satisfying*

(1) *For all integers $m > 0$, $h^0(M_\infty, K_{M_\infty}^{-m}) = h^0(M_i, K_{M_i}^{-m}) = 1 + \dfrac{m(m+1)}{2}(9-n)$.*

(2) *M_∞ has finitely many isolated singularities. Each of these singularities is either a rational double point (cf. [BPV], p87) or a singular point of type $C^2/Z_{l,p,q}$ with a cyclic group $Z_{i,p,q}$ defined in Lemma 5.5. Moreover, $lp^2 < \dfrac{48}{9-n}$. The latter singular point is a Hirzebruch-Jung singularity (cf. [BPV], p80).*

Proof. By Proposition 4.2, Lemma 5.2 and Lemma 5.5, it suffices to prove the upper bound of lp^2 required in the above (2).

By Bonnet-Myers Theorem ([CE]), we have

$$\text{diameter of } (M_i, g_i) \leqq \sqrt{3\pi} \text{ for all } i \tag{5.27}$$

It follows

$$\text{diameter of } (M_\infty, g_\infty) \leqq \sqrt{3\pi}. \tag{5.28}$$

For any fixed i and $x_i \in M_i$, by Bishop's Volume Comparison Theorem [Bi], we have

$$\frac{\mathrm{Vol}_{g_i}(B_r(x_i, g_i))}{9\mathrm{Vol}(B_{\frac{r}{\sqrt{3}}}(o, g_{S^4}))} > \frac{\mathrm{Vol}_{g_i}(M_i)}{9\mathrm{Vol}(S^4)} \text{ for } r \text{ small} \tag{5.29}$$

where (S^4, g_4) is the sphere with standard metric g_{S^4}, and o is the north pole in S^4.

Taking x_i in M_i such that $\lim_{i\to\infty} x_i = x_\infty$ is a singular point of type $C^2/Z_{l,p,q}$, then it follows from (5.29) that

$$\frac{1}{lp^2} > \frac{(9-n)}{18\mathrm{Vol}(S^4)} = \frac{9-n}{48}$$

i.e.

$$lp^2 < \frac{48}{9-n}. \tag{5.30}$$

The theorem is proved.

Remark.

(i) In case $n = 5$, there are three possible (l, p, q) for $Z_{l,p,q}$ in Theorem 5.1, i.e., $(l, p, q) = (1, 2, 1), (2, 2, 1), (1, 3, 1)$. Note that $Z_{(1,3,1)} \cong Z_{(1,3,2)}$.

(ii) In case $n = 6$, there are four possibilities: $(l, p, q) = (1, 2, 1), (2, 2, 1), (3, 2, 1), (1, 3, 1)$.

(iii) In case $n = 7$, the triple (l, p, q) could be $(l, 2, 1)$ for $1 \leqq l \leqq 5$, $(l, 3, 1)$ for $1 \leqq l \leqq 2$ and $(1, 4, 1)$.

(iv) In case $n = 8$, the triple (l, p, q) could be $(l, 2, 1)$ for $1 \leqq l \leqq 11$, $(l, 3, 1)$ for $1 \leqq l \leqq 5$, $(l, 4, 1)$ for $1 \leqq l \leqq 2$, $(1, 5, 1)$, $(1, 5, 2)$, and $(1, 6, 1)$.

6. Anticanonical divisors on some Kähler-Einstein orbifolds

We still denote by (M_∞, g_∞) the irreducible Kähler-Einstein orbifold in Proposition 4.2. Then this M_∞ is a normal surface with finitely many singular points. Each of these singularities is either a rational double point or a Hirzebruch-Jung singularity (cf. [BPV]). The purpose of this section is to study the plurianticanonical divisors on M_∞. Although the results here should hold in more general situation, we will confine our discussions to our special case.

Lemma 6.1. (Poincaré Duality Formula) *Let (M_∞, g_∞) be as above and ω_{g_∞} be the Kähler form associated to the metric g_∞. Then*

(1) *For any pluri-anti-canonical section $S \in H^0(M_\infty, K_{M_\infty}^{-m})$, we have*

$$\int_{\{S=0\}} \omega_{g_\infty}^2 = m \int_{M_\infty} \omega_{g_\infty}^2 = (9-n)m \tag{6.1}$$

(2) *Let D be a divisor in M_∞, $S \in H^0(M_\infty, K_{M_\infty}^{-m})$ and D_S is the divisor defined by the section S such that D and D_S have no common component, then*

$$m \int_D \omega_{g_\infty} = \sum_{x \in M_\infty} \frac{1}{\deg(\pi_x)} i_o(\pi_x^* D, \pi_x^* D_S) \tag{6.2}$$

where $\pi_x : \tilde{U}_x \to M_\infty$ is a local uniformization of x with $\pi_x(o) = x$ and $i_o(\pi_x^* D, \pi_x^* D_S)$ is the intersection multiplicity of $\pi_x^* D$ and $\pi_x^* D_S$ at the origin (cf. [BPV]). Note that x is smooth iff $\deg(\pi_x) = 1$.

Proof. The proof is standard. For example, in the case (1),

$$m \int_{M_\infty} \omega_{g_\infty}^2 = \lim_{\varepsilon \to 0} \int_{M_\infty} \omega_{g_\infty} \wedge \left(m\omega_{g_\infty} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\varepsilon + \|S\|_{g_\infty}^2) \right).$$

One can easily check that the right-handed side is just $\int_{\{S=0\}} \omega_{g_\infty}$. The case (2) can be similarly proved.

Let $\pi : \tilde{M}_\infty \to M_\infty$ be the minimal resolution. Then for each singular point x in M_∞, the exceptional curve $\pi^{-1}(x)$ is either an $A-D-E$ curve or a Hizebruch-Jung string according to whether x is a rational double point or not (cf. [BPV]). In particular, any singular point is rational. It is easy to show that $h^0(\tilde{M}_\infty, K_{\tilde{M}_\infty}^{-m}) = h^0(M_\infty, K_{M_\infty}^{-m})$ for any m. For any integer $m > 0$,

$$K_{\tilde{M}_\infty}^{-m} = \tilde{B}_m + \tilde{D}_m, \quad K_{M_\infty}^{-m} = B_m + D_m \tag{6.3}$$

where $\tilde{B}_m$(resp. B_m) consists of all one dimensional components in the base locus of $K_{\tilde{M}_\infty}^{-m}$(resp. $K_{M_\infty}^{-m}$). Then $\pi(\tilde{B}_m) = B_m$, $\pi(\tilde{D}_m) = D_m$. We denote by $|\tilde{D}_m|$ the linear system of the divisor $\tilde{D}_m$, etc.

We will first prove that the generic divisor in $|\tilde{D}_1|$ is irreducible. If $n = 8$, then $\dim|\tilde{D}_1| = h^0(\tilde{M}_\infty, \tilde{D}_1) - 1 = 1$, it follows that the generic divisor in $|D_1|$ is irreducible.

Lemma 6.2. *Let $n = 5, 6, 7$. Then if the generic divisor in $|\tilde{D}_1|$ is reducible, we can write $\tilde{D}_1 = (9-n)E$ with $E^2 = 0$ and $h^0(\tilde{M}_\infty, E) = 2$.*

Proof. We can write $\tilde{D}_1 = \sum_{i=1}^{l} \alpha_i E_i$ such that E_i is not linearly equivalent to E_j for $i \neq j$, any $|E_i|$ is free of one dimensional component in its base locus and the addition map: $\prod_{i=1}^{l} |E_i|^{\alpha_i} \to |\tilde{D}_1|$ is generically surjective. We need to prove that $l = 1$. Suppose that $l \geqq 2$. We may assume that E_1 can not be decomposed into the sum of E_2 and an effective divisor. We remark that the addition map: $|E_1| \times |E_2| \to |E_1 + E_2|$ is also generically surjective. Choose an irreducible divisor in $|E_1|$, say E_1 for simplicity. Then for any point x in E_1 outside the base locus of $|E_1 + E_2|$, we have a divisor in $|E_2|$ intersecting E_1 at x. Thus $E_1 \cdot E_2 \geqq 1 + \#\{\text{base points of } |E_1 + E_2|\}$ (count multiplicity). By Bertini's theorem (cf. [GH]), the generic divisor E in $|E_1 + E_2|$ is smooth outside the base locus of $|E_1 + E_2|$. So E can not be written as the sum of divisors in $|E_1|$ and $|E_2|$. A contradiction. Thus $l = 1$, $\tilde{D}_1 = \alpha E$ ($\alpha \geqq 2$). The above arguments also show that $h^0(\tilde{M}_\infty, E) = 2$. Since $h^0(\tilde{M}_\infty, \tilde{D}_1) = 10 - n$ and the generic divisor in $|\tilde{D}_1|$ can be written as the sum of α divisor in $|E|$, we must have $\alpha = 9 - n$. Then by Riemann-Roch Theorem [GH], $10 - n \geqq 1 + \alpha^2 E^2$, so $E^2 = 0$. The lemma is proved.

Denote by $IP(l, p, 1)$ be the germ of all holomorphic functions f at the origin of C^2 such that $\sigma_{l,p,1}^* f = e\left(\frac{1}{p}\right) f$. One can easily compute

$$IP(l, p, 1) = \{f \in C^2\{z_1, z_2\} \mid f = \sum_{k=0}^{pl-2} \left(z_1^k \sum_{j=0}^{\infty} f_j^k(z_1^{pl-1}, z_2) \right) \tag{6.4}$$

f_j^k are homogeneous polynomials of degree $j_k + jp^2 l\}$

where $j_0 = (p-1)pl$, $j_{k+1} \equiv j_k + (pl+1) \pmod{p^2 l}$ and $0 < j_k < p^2 l$ *for* $0 \leqq k \leqq pl - 2$.

The significance of $IP(l, p, 1)$ in our context is the following. If x is a singularity in M_∞ of type $C^2/Z_{l,p,1}$, let $\pi_x : \tilde{U}_x \to U_x \subset M_\infty$ be a local uniformization, where $\tilde{U}_x \subset C^2$, $\tilde{U}_x(o) = x$, then the local holomorphic sections of $K_{M_\infty}^{-1}$ in U_x correspond to one-to-one the functions in $IP(l, p, 1)$.

We list some simple lemmas in the following.

Lemma 6.3. *Let* $IP(l, p, 1)$ *be defined as in* (6.4). *Then*

(1) *The monomials in* $IP(l, 2, 1)$ *of degree* $\leqq 2l$ *are* $z_1^{2l}, z_2^{2l}, z_1 z_2, z_1^3 z_2^3, \ldots, z_1^{2k+1} z_2^{2k+1}$, *where* $k = \left[\frac{l-1}{2}\right]$.

(2) *If* $l = 1, 2$, *then the monomials in* $IP(l, 3, 1)$ *of degree* $\leqq \frac{9l}{2}$ *are among* $z_1^{3l}, z_1 z_2, z_1^4 z_2^4$.

(3) *The monomials in* $IP(1, 4, 1)$ *of degree* $\leqq 8$ *are* $z_1 z_2, z_2^4, z_1^2 z_2^6$.
In particular, if $p^2 l < 24$, *then for each* (λ, μ) *being either* $(4, 0)$, *or* $(0, 4)$ *or* $\lambda \geqq 2$, $\mu \geqq 2$, *there are at most two monomials in* $IP(l, p, 1)$ *with degree* $\leqq \frac{p^2 l}{2}$ *and containing the factor* $z_1^\lambda z_2^\mu$.

This follows directly from (6.4) and some simple computations.

Lemma 6.4. *Let f, g be holomorphic functions at $o \in C^2$ and have no common component. We further assume that $\sigma^*_{l,p,1} f = cf$ and $\{f = 0\}$ is smooth at 0. Then we have*

$$i_o(\{f = 0\}, \{g = 0\}) \geqq \begin{cases} \inf\{\lambda(p^2 l - pl + 1) + \mu | z_1^\lambda z_2^\mu \text{ is in } g\}, & \text{if } f = z_1 + O(|z|^2) \\ \inf\{\lambda + \mu(pl - 1) | z_1^\lambda z_2^\mu \text{ is in } g\}, & \text{if } f = z_2 + O(|z|^2) \end{cases} \tag{4.13}$$

where $i_o(\cdot, \cdot)$ is the intersection multiplicity (cf. [BPV]) and $|z|^2 = |z_1|^2 + |z_2|^2$.

Proof. We assume that $f = z_1 + O(|z|^2)$. The proof for the other case is same. Write $f = z_1 f_1(z_1, z_2) + f_2(z_2)$, then $f_1(0, 0) \neq 0$ and $\mathrm{ord}_o(f_2) \geqq 2$. By the assumption that $\sigma^*_{l,p,1} f = cf$, we have $\mathrm{ord}_o(f_2) \geqq p^2 l - pl + 1$. Put $w_1 = z_1 f_1(z_1, z_2) + f_2(z_2)$, $w_2 = z_2$, then $z_1 = w_1 \tilde{f}_1(w_1, w_2) + \tilde{f}_2(w_2)$ with $\mathrm{ord}_o(\tilde{f}_2) \geqq p^2 l - pl + 1$. Now by the definition of the intersection multiplicity, we have

$$i_0(\{f = 0\}, \{g = 0\}) = \mathrm{ord}_o(g|_{\{f=0\}}) = \mathrm{ord}_o(g(\tilde{f}_2(w_2), w_2)) .$$

Then lemma is proved.

We now come to apply these two lemmas to studying the properties the anticanonical divisors on M_∞.

Lemma 6.5. *Let $\tilde{D}_1$ be defined as in (6.3). Then the generic divisor in the linear system $|\tilde{D}_1|$ is irreducible unless $n = 7$, $B_1 = \varnothing$ and M_∞ has exactly two singular points of type $C^2/Z_{1,2,1}$.*

Proof. By some results in [De], we may assume that $n \leqq 7$ and M_∞ has at least one singular point besides rational double points. Let $x_1, \ldots, x_l$ ($l \geqq 1$) be those singular points in M_∞ other than rational double points, and $F_1, \ldots, F_l$ be the exceptional curves in the minimal resolution $\tilde{M}_\infty$ over $x_1, \ldots, x_\infty$. Suppose that the generic divisor in $|\tilde{D}_1|$ is reducible. By Lemma 6.2, we write $\tilde{D}_1 = (9 - n)E$.

Claim 1. The divisor E must intersect one of F_j, say F_1 for simplicity. Moreover, $E \cdot F_1 = 1$.

If $E \cdot F_1 = 0$ for any j, let H be a divisor of $K^{-m}_{M_\infty}$ for some $m > 0$ such that $x_j \notin H$ ($j = 1, 2, \ldots, l$). Then $H \cdot \pi(E) = K^{-m}_{M_\infty} \cdot E \geqq m$. By Lemma 6.1 (1), we have

$$\begin{aligned} 9 - n &= \int_{B_1} \omega_{g_\infty} + (9 - n) \int_{\pi(E)} \omega_{g_\infty} \\ &= \int_{B_1} \omega_{g_\infty} + \frac{(9 - n) H \cdot \pi(E)}{m} . \end{aligned}$$

Therefore, $B_1 = \varnothing$. It follows that all x_j are rational double points. A contradiction. So we may assume that $E \cdot F_1 \geqq 1$. Let F_{11} be the component in F_1 with $F_{11} \cdot E \geqq 1$. In case that $9 - n \geqq 3$ or there is another component in $\tilde{B}_1$ intersecting F_{11} at some point, the divisor F_{11} has multiplicity two in $\tilde{B}_1$ by $-K_{\tilde{M}_\infty} \cdot F_{11} = F_{11}^2 + 2$. It follows that $E \cdot F_{11} + E \cdot F_1 \leqq \tilde{B}_1 \cdot E \leqq 2$, i.e. $E \cdot F_1 = 1$. Thus we may assume that x_1 is of type $C^2/Z_{1,2,1}$ and $F_1 \cdot (\tilde{B}_1 - F_1) = 0$. By adjunction formula, we have $-2 = K^1_{\tilde{M}_\infty} \cdot F_1 \geqq F_1^2 + 2F_1 \cdot E = -4 + 2F_1 \cdot E$, i.e., $F_1 \cdot E = 1$. The claim is proved.

Claim 2. If x_1 is of type $C^2/Z_{1,2,1}$, then $n = 7$, $B_1 = \varnothing$ and M_∞ has exactly two singular points of type $C^2/Z_{1,2,1}$.

We first prove that E must intersect another exceptional curve F_j other than F_1. In fact, if it is not true, then F_1 has multiplicity two in $\tilde{B}_1$. By $-K_{\tilde{M}_\infty^*} \cdot F_1 = F_1^2 + 2$, there are exactly six curves in a generic anticanonical divisor on $\tilde{M}_\infty$ intersecting F_1. Thus for any S in $H^0(M_\infty, K_{M_\infty}^{-1})$, the pull-back $\pi_1^* S$ is locally represented by a holomorphic function f_S in $IP(1, 2, 1)$ with vanishing order at least 6 at o, where $\pi_1 : \tilde{U}_1 \to U_1$ is the local uniformization of x_1 in M_∞. By (6.2) in Lemma 6.1, if $\{S = 0\}$ has no common component with $\pi(E)$, we have

$$9 - n = \int_{B_1} \omega_{g_\infty} + (9 - n) \int_{\pi(E)} \omega_{g_\infty} \geqq \frac{(9-n)}{4} \operatorname{mult}_o(\{S = 0\}) \geqq \frac{3}{2}(9 - n) .$$

A contradiction. Therefore, the divisor E intersects with another exceptional curve, say F_2. Then $n = 7$ and $F_2 \cdot E = 1$, so by $K_{\tilde{M}_\infty^*}^{-1} \cdot F_2 = F_2^2 + 2$, the singular point x_2 must be of type $C^2/Z_{1,2,1}$. As above, by using Lemma 6.1, one can prove that $B_1 = \varnothing$. Thus there is no other singular point in M_∞ besides rational double points. Claim 2 is proved.

Now we may assume that x_1 is not of type $C^2/Z_{1,2,1}$, then F_1 has multiplicity two in $\tilde{B}_1$ and E intersects with no other $F_j (j \geqq 2)$. By (6.2) in Lemma 6.1, we have

$$\int_{\pi(E)} \omega_{g_\infty} \leqq 1 \quad \text{and} \quad < 1 \text{ if } B_1 \neq \varnothing . \tag{6.5}$$

Let $\pi_1 : \tilde{U}_1 \to U_1$ be the local uniformization of x_1 with uniformization group Γ_1, then as above, the section S in $H^0(M_\infty, K_{M_\infty}^{-1})$ is locally represented by a f_S in $IP(l, p, 1)$, where x_1 is of type $C^2/Z_{l,p,1}$. It follows from (6.5) and Lemma 6.1,

$$i_o(\{f_S = 0\}, \pi_1^{-1}(\pi(E))) \leqq p^2 l \quad \text{and} \quad < p^2 l \text{ if } B_1 \neq \varnothing \tag{6.6}$$

where $i_o(\cdot, \cdot)$ is the intersection multiplicity at the origin in $\tilde{U}_1$.

Claim 3. For generic E in the linear system $|E|$, the pull-back $\pi_1^{-1}(\pi(E))$ is smooth at the origin.

Let (z_1, z_2) be the local coordinates of $\tilde{U}_1$ such that the generator $\sigma_{l,p,1}$ of $\Gamma_1 \cong Z_{l,p,1}$ is diagonal in (z_1, z_2). Also let h_E be the defining function of $\pi_1^{-1}(\pi(E))$ in $\tilde{U}_1$, then $\sigma_{l,p,1}^* h_E = c h_E$ for some constant c. First we prove that $\operatorname{ord}_o(h_E) \leqq 2$ for generic E. If this is not true, then $\operatorname{ord}_o(f_S) \geqq 6$ for any section S in $H^0(M_\infty, K_{M_\infty}^{-1})$. By Lemma 6.3 and the fact that $h^0(M_\infty, K_{M_\infty}^{-1}) \geqq 3$, one can easily find a section S in $H^0(M_\infty, K_{M_\infty}^{-1})$, the local representation f_S vanishes at o of order $> \dfrac{p^2 l}{3}$ for those $(l, p, 1)$ with $p^2 l < 24$. Now we choose a generic E such that E has no common component with $\{S = 0\}$, then we have

$$i_o(\{f_s = 0\}, \pi_1^{-1}(\pi(E))) \geqq \operatorname{ord}_o(h_E) \cdot \operatorname{ord}_o(f_S) > p^2 l .$$

It contradicts to (6.6). So $\operatorname{ord}_o(h_E) \leqq 2$. Using the invariance $\sigma_{l,p,1}^* h_E = c h_E$ and the fact that $\sigma_{l,p,1}$ has distinct eigenvalues, we conclude that $h_E(z_1, z_2) = z_j^\lambda + O(|z|^{\lambda+1})$ or $z_1 z_2 + O(|z|^3)$, where $\lambda = 1, 2$. In the former case, if $\lambda = 2$, then for any S in $H^0(M_\infty, K_{M_\infty}^{-1})$, the lowest order term of f_S has the fact z_j^4, so

by Lemma 5.3 again, we can find such a S that $\mathrm{ord}_o(f_S) > \frac{p^2 l}{2}$. As above, we can deduce a contradiction to (6.6). Some arguments exclude the possibility of $h_E = z_1 z_2 + O(|z|^3)$. Therefore $\lambda = 1$ and the claim is proved.

We now want to apply the formula in Lemma 6.4 to our cases and conclude the proof of this lemma. It suffices to find a S in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that for any term $z_1^\lambda z_2^\mu$ in f_S, either $\lambda(p^2 l - pl + 1) + \mu > p^2 l$ for $j = 1$, where $h_E = z_1 + O(|z|^2)$, or $\lambda + \mu(pl - 1) > p^2 l$ for $j = 2$, where $h_E = z_2 + O(|z|^2)$. Since there is only one monomial $z_1^\lambda z_2^\mu$ in $IP(l, p, 1)$ such that $\lambda + \mu \leqq p^2 l$ and $\lambda \leqq 1$, we may find a S in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that any term $z_1^\lambda z_2^\mu$ in f_S has either $\lambda + \mu > p^2 l$ or $\lambda \geqq 2$. So by the fact that $2(p^2 l - pl + 1) > p^2 l$ for $p \geqq 2$, we may assume that $h_E = z_2 + O(|z|^2)$. By (6.4), any monomial $z_1^\lambda z_2^\mu$ in $IP(l, p, 1)$ can be written $z_1^{k+(pl-1)\lambda'} z_2^\mu$ with $\lambda' + \mu = j_k$, thus

$$\lambda + \mu(pl - 1) = k + (pl - 1)j_k .$$

One can easily check that for those $(l, p, 1)$ with $p^2 l < 24$, the monomials $z_1 z_2$, z_1^{pl} are only ones in $IP(l, p, 1)$ with $\lambda + \mu(pl - 1) \leqq p^2 l$. Then by the fact that $h^0(M_\infty, K_{M_\infty}^{-1}) \geqq 3$, we may find a S in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that f_S does not contain $z_1 z_2$ and z_1^{pl}. It follows from Lemma 6.4,

$$i_0(\{h_E = 0\}, \{f_S = 0\}) > p^2 l \tag{6.7}$$

Therefore, we obtain a contradiction from (6.6) and (6.7). The lemma is proved.

Proposition 6.1. *Let $\{(M_i, g_i)\}, (M_\infty, g_\infty)$ be given as in Proposition 4.2. Assume that $n = 7$. Then M_∞ has only rational double points as singular points unless M_∞ has exactly two singular points of type $C^2/Z_{1,2,1}$, and $|K_{M_\infty}^{-1}|$ is free of one dimensional components in its base locus. Moreover, the linear system $|K_{M_\infty}^{-2}|$ is always free of base point.*

Proof. It is well-known that each M_i is branched double covering of CP^2, in particular, each M_i admits a nontrivial involution τ_i (cf. [De]). One can easily check that the fixed point set A_i of τ_i is a connected smooth divisor in $|2K_{M_i}^{-1}|$ and τ_i preserves any anticanonical divisor. These τ_i converge to a nontrivial involution τ_∞ of M_∞ as M_i converge to M_∞ in the sense of Proposition 4.2. The fixed point set A_∞ of τ_∞ is the limit of A_i and then is 2-anticanonical divisor in $|2K_{M_\infty}^{-1}|$.

Let $\pi: \tilde{M}_\infty \to M_\infty$ be the minimal resolution as above and $\tilde{B}_1$, $\tilde{D}_1$, B_1, D_1 defined as in (6.3). We first assume that the generic divisor in $|D_1|$ is irreducible. Choose such an irreducible one, say D_1 for simplicity. Fix a regular point x of M_∞ in $D_1 \setminus (A_\infty \cup B_1)$. Since $h^0(M_\infty, K_{M_\infty}^{-1}) = 3$, we can find another divisor D_1' in $|D_1|$ such that D_1 and D_1' have no common component and $x \in D_1 \cap D_1'$. Since both D_1 and D_1' are stabilized by τ_∞, their intersection $D_1 \cap D_1'$ also contains $\tau_\infty(x)$. By Lemma 6.1, we have

$$\begin{aligned} 2 &= \int_{B_1} \omega_{g_\infty} + \int_{D_1} \omega_{g_\infty} \\ &\geqq \int_{B_1} \omega_{g_\infty} + \sum_{y \in \mathrm{Sing}(M_\infty)} \frac{1}{\deg(\pi_y)} i_o(\pi_y^* D_1, \pi_y^* D_1') + i_x(D_1, D_1') + i_{\tau_\infty(x)}(D_1, D_1') \end{aligned}$$

where $\pi_y: \tilde{U}_y \to U_y$ is the local uniformization of y in the singular set $\mathrm{Sing}(M_\infty)$. It follows that $B_1 = \varnothing$ and none of singular points in M_∞ is in the base locus of $|K_{M_\infty}^{-1}|$. The latter implies that all singular points of M_∞ are rational double points (cf. [BPV]).

It remains to conside the case that the generic divisor in $|D_1|$ is reducible. Then by Lemma 6.5, it suffices to prove that the base locus of $|2K_{M_\infty}^{-1}|$ does not contain those points of type $C^2/Z_{1,2,1}$. Now we have a natural divisor A_∞ in $|2K_{M_\infty}^{-1}|$. It is smooth and irreducible since it is the fixed point set of τ_∞ and τ_∞ is an isometry of (M_∞, g_∞). We claim that A_∞ does not contain an singular point of type $C^2/Z_{1,2,1}$. In fact, if the claim is not true, both singular points p_1, p_2 of type $C^2/Z_{1,2,1}$ are in A_∞. By our assumption on $|D_1|$, we can write $D_1 = 2E$ and the generic divisor in $|E|$ is an irreducible rational curve passing through p_1, p_2. On the other hand, since τ_∞ preserves $|D_1|$, it also preserves $|E|$. Choose a generic divisor in $|E|$, say E for simplicity, such that E intersects with A_∞ at a point outside p_1, p_2. Note that two generic divisors in $|E|$ do not intersect to each other outside p_1, P_2. Thus τ_∞ must stabilize E, then it fixes E since it fixes three points on E and E is rational. It follows that τ_∞ fixes the generic points in M_∞, i.e., τ_∞ is an identity. A contradiction! Therefore, $p_1, p_2 \notin A_\infty$ and $|2K_{M_\infty}^{-1}|$ is free of base point. The proposition is proved.

Remark. One can also construct local nonvanishing sections of $2K_{M_\infty}^{-1}$ at the above p_1, p_2 and then use L^2-estimate of $\bar{\partial}$-operators (Proposition 5.1) with weight function $a\log(\sum_{\beta=0}^{2} \|S_\beta^\infty\|_{g_\infty}^2)$ to produce a section of $|2K_{M_\infty}^{-1}|$ which is nonzero at p_1, p_2, where $\{S_\beta^\infty\}_{0 \le \beta \le 2}$ is an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-1})$ with respect to the inner product induced by g_∞. In particular, it implies that $|2K_{M_\infty}^{-1}|$ is free of base point.

Proposition 6.2. *Let $\{(M_i, g_i)\}$ and (M_∞, g_∞) be as in Proposition 4.2. Assume that $n = 8$. Then M_∞ has at most one singular point of type $C^2/Z_{l,2,1}$ $(2 \le l \le 7)$ besides rational double points. Moreover, the linear system $|2K_{M_\infty}^{-1}|$ is free of base point.*

Proof. It is known (cf. [De]) that each M_i is a branched double covering over a quadratic cone in CP^3. It implies that each M_i admits a nontrivial involution τ_i. Also, this τ_i preserves both anticanonical divisors and 2-anticanonical divisors. These τ_i converge to a nontrivial involution τ_∞ of M_∞ as M_i converge to M_∞ in the sense of Proposition 4.2.

Let $\tilde{M}_\infty$ be minimal resolution of M_∞ and $\tilde{B}_m$, $\tilde{D}_m$, B_m, D_m be defined as in (6.3). The involution τ_∞ can be lifted to $\tilde{M}_\infty$, still denoted by τ_∞ for simplicity (cf. [La]). Then τ_∞ stabilizes B_m, D_m, $\tilde{B}_m$, $\tilde{D}_m$ $(m = 1, 2)$, respectively. We assume that M_∞ has a singular point other than rational double points.

Claim 1. $\tilde{D}_1^2 = 0$

By Riemann-Roch Theorem and the fact that $h^0(\tilde{M}_\infty, \tilde{D}_1) = 2$, we have either $\tilde{D}_1^2 = 0$ or $\tilde{D}_1^2 = 1$ and $\tilde{B}_1 \cdot \tilde{D}_1 = 0$. If $\tilde{D}_1^2 = 1$, then there are two irreducible divisors $\tilde{D}'_1$, $\tilde{D}''_1$ in $|\tilde{D}_1|$ such that $\tilde{D}'_1$ intersects $\tilde{D}''_1$ at a point outside $\tilde{B}_1$. By Lemma 6.1,

$$1 = \int_{B_1} \omega_{g_\infty} + \int_{D_1} \omega_{g_\infty} \ge \int_{B_1} \omega_{g_\infty} + 1$$

It follows that $B_1 = \varnothing$. Thus all singular points are rational double points. A contradiction. The claim is proved.

The same arguments also prove that D_1 does not intersect B_1 outside singular points. Now $|\tilde{D}_1|$ induces a fibration $\pi_f: \tilde{M}_\infty \to CP^1$. We claim that the generic divisor in $|\tilde{D}_2|$ has a horizontal component. In fact, if it is not true, then by $h^0(\tilde{M}_\infty, \tilde{D}_2) = 4$, we have $\tilde{D}_2 = 3\tilde{D}_1$. It follows that $\tilde{D}_1 + \tilde{B}_2 = 2\tilde{B}_1$. Since $\tilde{B}_2 \subset 2\tilde{B}_1$, the divisor $2\tilde{B}_1 - \tilde{B}_2$ is effective. By $\tilde{D}_1^2 = 0$, we conclude that $2\tilde{B}_1 - \tilde{B}_2$ is vertical with respect to the fibration $\pi_f: \tilde{M}_\infty \to CP^1$. It is easy to prove that $2\tilde{B}_1 - \tilde{B}_2$ is connected since $\tilde{D}_1$ is. Let E be the reduced divisor supporting $2\tilde{B}_1 - \tilde{B}_2$, then E is a proper subset of one fiber in $\tilde{M}_\infty$ and $2\tilde{B}_1 - \tilde{B}_2 - E = \tilde{D}_1 - E$. Since $2\tilde{B}_1 - B_2 - E$ and $\tilde{D}_1 - E$ have no cominon irreducible component, it follows that $(\tilde{D}_1 - E)^2 \geqq 0$. On the other hand, the divisor $\tilde{D}_1 - E$ is a proper subset of a fiber, then $(\tilde{D}_1 - E)^2 < 0$. We get a contradiction. Therefore, the claim is proved.

Now choose a generic $\tilde{D}_2$ in $|\tilde{D}_2|$ such that the generic divisor in $|\tilde{D}_1|$ intersects $\tilde{D}_2$ at at least one point outside $\tilde{B}_1$. Since τ_∞ preserves divisors in $|\tilde{D}_1|$ and $|\tilde{D}_2|$, we have that $\pi(\tilde{D}_2)$ intersects the generic divisor in $|D_1|$ at at least two smooth points in M_∞. By Lemma 6.1, we can conclude that $B_1 = \varnothing$ and $\pi(\tilde{D}_2)$ does not pass through the singularities of M_∞ other than rational double points. Then there are at most two singular points of type $C^2/Z_{l,2,1}$ in M_∞ besides rational double points. By adjunction formula, one can easily prove that there is at most one singular point in M_∞ besides rational double points. The above arguments also show that $|2K_{M_\infty}^{-1}|$ is free of base point.

Corollary 6.1. *Let $n = 7$ or 8. There are constants $c(n, k)$ depending only on $n, k \geqq 1$ such that for any Kähler-Einstein surface (M', g') in $\mathfrak{I}_n$, we have*

$$\inf_{M'} \left\{ \sum_{\beta=0}^{N_m} \|S'_\beta\|^2_{g'} \right\} \geqq c(n, k) \tag{6.8}$$

where $m = 2k, N_m + 1 = \dim_C H^0(M', K_{M'}^{-m})$ and $\{S'_\beta\}_{0 \leq \beta \leq N_m}$ is an orthonormal basis of $H^0(M', K_{M'}^{-m})$ with respect to the inner product induced by g'.

Proof. It follows from Lemma 5.3, Propositions 6.1 and 6.2 (cf. the proof of Theorem 5.1).

7. Completion of the proof for strong partial C^0-estimate

In this section, we will complete the proof of Theorem 2.2, i.e., the strong partial C^0-estimate stated in section 2. By Corollary 6.1, Lemma 5.3 and the arguments in the proof of Theorem 5.1, Theorem 2.2 will follows from the following proposition.

Proposition 7.1. *Let $n = 5$ or 6, and (M_∞, g_∞) be the irreducible Kähler-Einstein orbifold in Proposition 4.2 or Theorem 5.1. Then the linear system $|6K_{M_\infty}^{-1}|$ is free of the base point.*

The rest of this section is devoted to the proof of this proposition. The basic tools are still the Poincaré duality formula (Lemma 6.1) and adjunction formula we have used in last section.

We fix the Kähler-Einstein orbifold (M_∞, g_∞) as in Proposition 7.1 with $C_1(M_\infty)^2 = 9 - n$, where $n = 5$ or 6. Let $\pi: \tilde{M}_\infty \to M_\infty$ be the minimal resolution and $x_1, \ldots, x_k$ be the singular points of M_∞ besides rational double points. The corresponding exceptional divisors in M_∞ are denoted by $F_1, \ldots, F_k$. Define D_m, B_m, $\tilde{D}_m$, $\tilde{B}_m$ as in (6.3). If M_∞ has only rational double points as singular points, then by the results in [De], the linear system $|K_{M_\infty}^{-1}|$ is free of base point. So we suppose that $k \geqq 1$. As in the proof of Proposition 6.1, etc, we may assume that F_1 intersects $\tilde{D}_1$. We collect some simple facts either built up before or that can be easily proved by using Riemann-Roch Theorem and adjunction formula (cf. [BPV], [GH]).

(F1) The generic divisor in $|\tilde{D}_1|$ is a smooth rational curve and $\tilde{D}_1^2 = 8 - n$, $K_{\tilde{M}_\infty}^{-1} \cdot \tilde{D}_1 = \tilde{D}_1^2 + 2$.

(F2) For $8 - n$ generic distinct points $\{y_j\}_{1 \leqq j \leqq 8-n}$ in $\tilde{M}_\infty$ outside exceptional curves, there is a pencil of divisors in $|\tilde{D}_1|$, denoted by $|(\tilde{D}_1, \{y_j\}_{1 \leqq j \leqq 8-n})|$ such that the generic divisor in this pencil is a smooth rational curve and $|(\tilde{D}_1, \{y_j\}_{1 \leqq j \leqq 8-n})|$ is free of base point outside $\{y_j\}_{1 \leqq j \leqq 8-n}$.

(F3) Let E be an exceptional, irreducible curve, i.e. $E^2 < 0$, then if $E \notin \tilde{B}_1$ and $\tilde{B}_1 \cdot E > 0$, then E is of first kind, i.e. $E^2 = -1$. Moreover, this E intersects exactly one irreducible component in $\tilde{B}_1$.

(F4) Each singular point $x_j (1 \leqq j \leqq k)$ is of type either $C^2/Z_{l,2,1} (1 \leqq l \leqq 3)$ or $C^2/Z_{1,3,1}$. Thus the corresponding exceptional curve F_j is reduced and can be written as $F_{j1} + \ldots + F_{jk_j}$ such that $F_{ji} \cdot F_{ji'} = 1$ if $|i - i'| = 1$; $= 0$ if $|i - i'| > 1$ and F_{ji} is irreducible for each i between 1 and k_j, where $k_j = 1, 2, 3$. If $k_j = 1$, then $F_{ji}^2 = -4$ and x_j is of type $C^2/Z_{1,2,1}$. If $k_j = 2$, then either $F_{j1}^2 = F_{j2}^2 = -3$ or $F_{j1}^2 = -2$, $F_{j2}^2 = -5$ according to that x_j is of type either $C^2/Z_{2,2,1}$ or $C^2/Z_{1,3,1}$. If $k_j = 3$, then x_j is of type $C^2/Z_{3,2,1}$, and $F_{j1}^2 = F_{j3}^2 = -3$, $F_{j2}^2 = -2$.

By these facts, there are three cases of $\tilde{M}_\infty$ as follows.

Case 1. $F_1 \cdot \tilde{D}_1 = 2$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 2$.

Case 2. $F_1 \cdot \tilde{D}_1 = 1$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 2$, so the irreducible component in F_1 intersecting $\tilde{D}_1$ has multiplicity two in $\tilde{B}_1$.

Case 3. $F_1 \cdot \tilde{D}_1 = 1$ and $F_2 \cdot \tilde{D}_1 = 1$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 3$.

We will treat these cases separately in the following lemmas.

Lemma 7.1. *Let (M_∞, g_∞), F_j, $\tilde{D}_1$, etc. be given as in Proposition 7.1 and $F_1 \cdot \tilde{D}_1 = 2$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 2$. Then $B_1 = \varnothing$, $\tilde{B}_1 = F_1$ and $\tilde{D}_1$ intersects with F_{11} and F_{1k_1} at one point, respectively. Moreover, the linear system $|6K_{M_\infty}^{-1}|$ is free of base point.*

Proof. Let α_i be the multiplicity of the irreducible component F_{1i} $(1 \leqq i \leqq k_1)$ in $\tilde{B}_1$, then by adjunction formula and (F1), we have

$$2 \geqq \tilde{B}_1 \cdot \tilde{D}_1 \geqq \alpha_i F_{1i} \cdot \tilde{D}_1 + (\tilde{B}_1 - F_{1i}) \cdot \tilde{D}_1$$

$$\tilde{D}_1 \cdot F_{1i} + (\tilde{B}_1 - F_{1i}) \cdot F_{1i} = 2 \tag{7.1}$$

It follows from (7.1) and $F_1 \cdot \tilde{D}_1 = 2$ that $\tilde{D}_1 \cdot \tilde{F}_{1i} \leqq 1$ unless $k_1 = 1$, i.e., x_1 is of type

$C^2/Z_{1,2,1}$, and $D_1 \cdot F_{1i} = 1$ iff $(\tilde{B}_1 - F_{1i}) \cdot F_{1i} = 1$ and $\alpha_i = 1$. Therefore $\tilde{D}_1 \cdot F_{1i} = 1$ for $i = 1, k_1$ and $(\tilde{B}_1 - F_1) \cdot F_1 = 0$, i.e., $x_1 \notin B_1$. By Lemma 6.1 and the above (F2), the generic divisor D_1 does not intersect B_1 outside $x_1, \ldots, x_k$. On the other hand, since (M_∞, g_∞) is a limit of some sequence of Kähler-Einstein surfaces according to Proposition 4.2, any anticanonical divisor in $|K_{M_\infty}^{-1}|$ must be connected, so $D_1 \cap B_1$ contains some x_j for $j \geqq 2$ if $B_1 \neq \varnothing$. By our assumption, we have $x_j \notin D_1$ for $j \geqq 2$. It implies that $B_1 = \varnothing$ and $\tilde{B}_1 = F_1$.

It remains to prove that $|6K_{M_\infty}^{-1}|$ is free of base point. By Lemma 6.1 and the above (F2), one can easily prove that $|K_{M_\infty}^{-1}| = |D_1|$ is free of base point outside x_1. So we only need to construct a global section of $6K_{M_\infty}^{-1}$ not vanishing at x_1. It will be done by applying Proposition 5.1. Define

$$\psi = 6\log\left(\sum_{\beta=0}^{N_1} \|S_\beta^\infty\|_{g_\infty}^2\right) \tag{7.2}$$

where $N_1 = \dim_C H^0(M_\infty, g_\infty) - 1$ and $\{S_\beta^\infty\}$ is an orthonormal basis with respect to the inner product on $H^0(M_\infty, K_{M_\infty}^{-1})$ induced by g_∞. Since the base locus of $K_{M_\infty}^{-1}$ is the point x_1, the function ψ is bounded and continuous outside x_1. As we remarked in §6, each section S_β^∞ is represented by a function in $IP(l, p, 1)$ in the local uniformization of x_1, where x_1 is of type $C^2/Z_{l,p,1}$. In particular, it follows that for any neighborhood U of x_1 in M_∞, we have

$$\int_U e^{-\psi} dV_{g_\infty} = +\infty \tag{7.3}$$

By the definition of ψ in (7.2), for any tangent vector v of type (1.0) at any point of X, we have

$$\left\langle \partial\bar{\partial}\psi + \frac{14\pi}{\sqrt{-1}}\omega_{g_\infty}, v \wedge \bar{v} \right\rangle_{g_\infty} \geqq \|v\|_{g_\infty}^2 \tag{7.4}$$

Thus by Proposition 5.1, in order to have a global section of $6K_{M_\infty}^{-1}$ nonvanishing at x_1, it suffices to construct a nonvanishing local section of $6K_{M_\infty}^{-1}$ in neighborhood of x_1. It is obviously possible since x_1 is a singular point of type $C^2/Z_{l,p,1}$ with $1 \leqq l \leqq 3$, $2 \leqq p \leqq 3$. Then the lemma is proved.

Lemma 7.2. *Let (M_∞, g_∞), F_j, $\tilde{D}_1$, etc. be given as in Proposition 7.1. Then the irreducible component in F_1 intersecting $\tilde{D}_1$ has the multiplicity one in $\tilde{B}_1$.*

Proof. We prove it by contradiction. Assume that the conclusion of this lemma is not true. First let F_1 be irreducible, i.e. x_1 is of type $C^2/Z_{1,2,1}$. Then by Lemma 4.10 and adjunction formula $K_{M_\infty}^{-1} \cdot F_1 = F_1^2 + 2$, there are at least five irreducible components $C_\alpha (1 \leqq \alpha \leqq 5)$ in $\tilde{B}_1$ such that $C_\alpha \cdot F_1 = 1$. Obverse that the pluriantanonical divisor $K_{M_\infty}^{-6}$ is an ample Cartier one, so

$$\sum_{\alpha=1}^{5} \int_{C_\alpha} \omega_{g_\infty} \geqq \sum_{\alpha=1}^{5} \frac{1}{6} = \frac{5}{6} \tag{7.5}$$

Choose two generic divisors $\tilde{D}_1, \tilde{D}_1'$ in $|\tilde{D}_1|$ such that $\tilde{D}_1$ intersects $\tilde{D}_1'$ at $8 - n$ distinct points outside $\tilde{B}_1$. Let S' be the section in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that

$\{S'=0\}$ is just the sum of $\pi(\tilde{D}'_1)$ and B_1. Thus one can compute that

$$\sum_{z\in M_\infty}\frac{1}{\deg(\pi_z)}i_o(\pi_z^{-1}\{S'=0\},\pi_z^{-1}(\pi(\tilde{D}_1)))\geqq 8-n+\frac{1}{2} \tag{7.6}$$

where $\pi_z:\tilde{U}_z\to U_z$ is the local uniformization of M_∞ at z with $\pi_z(o)=z$. Combining (7.6) with (7.5), we get a contradiction to Lemma 6.1. So x_1 can not be of type $C^2/Z_{1,2,1}$.

If x_1 is of type $C^2/Z_{2,2,1}$, then by the same arguments as above, we can prove that B_1 does not contain more than three irreducible components. Write $\tilde{B}_1=2F_{11}+\alpha F_{12}+\tilde{B}'_1$ where $\tilde{B}'_1$ is an effective divisor having no common component with F_1. Note that any irreducible component E in $\tilde{B}'_1$ with $E\cdot F_1>0$ can not be contracted to a point by the projection $\pi:\tilde{M}_\infty\to M_\infty$, in particular, $E^2=-1$. Moreover, if $\pi(E)$ does not pass through a singular point of type $C^2/Z_{1,3,1}$, then $\int_{\pi(E)}\omega_{g_\infty}\geqq\frac{1}{2}$ and B_1 contains at most one irreducible component. By adjunction formula, it contradicts to the fact that F_{11} has multiplicity two in $\tilde{B}_1$ and $F_1\cdot\tilde{D}_1=1$. So for any such an E, $\pi(E)$ must pass through a singular point of type $C^2/Z_{1,3,1}$. Then one can easily show that $\alpha=1$ and $B_1=3\pi(E)$, where E is an irreducible component intersecting F_{11}. Let $\pi_1:\tilde{U}_{x_1}\to U_{x_1}$ be the local uniformization of M_∞ at x_1. Then any section S in $H^0(M_\infty,K_{M_\infty}^{-1})$ is locally presented by a holomorphic function f_S on $\tilde{U}_{x_1}$ of form $(h_E)^3\tilde{f}_S$, where h_E is the defining function of $\pi_1^{-1}(E)$ in $\tilde{U}_{x_1}$ and $\deg_o(\tilde{f}_S)\geqq 1$, $\deg_o(h_E)\geqq 1$. Since f_S is also in $IP(2,2,1)$, there is at most one monomial term in f_S with degree less than 5. Thus we can choose two divisors $\tilde{D}_1$, $\tilde{D}'_1$ in $|\tilde{D}_1|$ such that $\pi(\tilde{D}_1)$ intersects $\{S'=0\}$ at $8-n$ distinct points outside B_1 and $\deg_o(f_{S'})\geqq 5$, where S' is the section in $H^0(M_\infty,K_{M_\infty}^{-1})$ with $\{S'=0\}=\pi(\tilde{D}'_1)\cup B_1$. By Lemma 6.1,

$$\begin{aligned}9-n&=3\int_{\pi(E)}\omega_{g_\infty}+\int_{\pi(\tilde{D}_1)}\omega_{g_\infty}\\&\geqq\frac{1}{2}+8-n+\frac{1}{\deg(\pi_1)}i_o(\pi_1^{-1}(\pi(\tilde{D}_1)),\{f_{S'}=0\})\\&\geqq\frac{1}{2}+8-n+\frac{5}{8}>9-n\end{aligned}$$

A contradiction! Thus x_1 is not of type $C^2/Z_{2,2,1}$, either.

Next, we assume that x_1 is of type $C^2/Z_{3,2,1}$, then $n=6$. Let $\tilde{B}_1=\alpha F_{11}+\beta F_{12}+\gamma F_{13}+\tilde{B}'_1$, where $F_{11}^2=F_{13}^2=-3$, $F_{12}^2=-2$ and $\tilde{B}'_1$ does not contain $F_{1j}(1\leqq j\leqq 3)$ any more.

Claim 1. The generic $\tilde{D}_1$ intersects F_{12}.

If the claim is not true, then we may assume that $\tilde{D}_1\cdot F_{11}=1$, so $\alpha=2,\beta\geqq 2$. If $\gamma=1$, then $\beta=2$. By adjunction formula, we have $\tilde{B}'_1\cdot F_{11}=2$, $\tilde{B}'_1\cdot F_{12}=1$, $\tilde{B}'_1\cdot F_{13}=0$. Let E' be the exceptional curve in $\tilde{B}'_1$ intersecting with F_{12}, then $E'\cdot(\tilde{B}_1-2F_{12})=0$. It follows that $\pi(E')$ does not pass through any singular point other than x_1 and rational double points, so $\int_{\pi(E')}\omega_{g_\infty}\geqq\frac{1}{2}$. By Lemma 6.1, it implies that $B_1=\pi(E')$ and then $\tilde{B}'_1=E'$. It contradicts to $\tilde{B}'_1\cdot F_{11}=2$. Thus $\gamma\geqq 2$. We observe that no E in $\tilde{B}'_1$ intersects two components of F_1 by adjunction formula or

Lemma 6.1. So if we let k_j be the number of irreducible components in $\tilde{B}'_1$ intersecting with $F_{1j}(1 \leqq j \leqq 3)$, then $k_1 + k_2 + k_3 \leqq 3$ by Lemma 6.1. By adjunction formula, we have

$$\begin{cases} 1 - 6 + \beta + k_1 = -1 \\ 2 - 2\beta + \gamma + k_2 = 0 \\ \beta - 3\gamma + k_3 = -1 \end{cases} \tag{7.7}$$

Summing these three equations, we derive $\gamma \leqq 1$. A contradiction. The claim is proved.

Then $\beta = 2$ and $\alpha + \gamma \leqq 3$. By the above arguments in excluding that x_1 is of type $C^2/Z_{2,2,1}$, one can prove that $\alpha = 1$, $\gamma = 1$. Thus $B_1 = F_{11} + 2F_{12} + F_{13} + E$, where E is an exceptional curve of first kind and $E \cdot F_{12} = 1$. There are now two methods to conclude a contradiction. One of them is to use Lemma 6.1. We can easily choose two divisors D_1, D'_1 in $|D_1|$ such that they intersect to each other exactly at $8 - n$ distinct points besides x_1 and $i_o(\pi_1^{-1}(D_1), \pi_1^{-1}(D'_1 + B_1)) \geqq 7$, where $\pi_1 : \tilde{U}_1 \to U_{x_1}$ is a local uniformization of x_1. It will contradict to formula (6.1) in Lemma 6.1. There is another method described as follows. Let $\tilde{M}_\infty^0$ be the surface obtained by blowing down E and then F_{12} in $\tilde{M}_\infty$, and F_{1j}^0 be the images of $F_{1j}(j = 1, 3)$ under the natural projection from $\tilde{M}_\infty$ onto $\tilde{M}_\infty^0$. Then $(F_{1j})^2 = -2(j = 1, 3)$. Inductively, let $\tilde{M}_\infty^k$ be the surface obtained by blowing down an exceptional curve intersecting either F_{11}^{k-1} or F_{13}^{k-1}. Put $F_{1j}^k = \pi'_k(F_{1j}^{k-1})(j = 1, 3)$, where $\pi'_k : \tilde{M}_\infty^{k-1} \to \tilde{M}_\infty^k$ is the natural projection. Let $\tilde{M}_\infty^m$ be the last surface obtained in such a process. Then $(K_{\tilde{M}_\infty^m})^2 = 4 + m$. By $h^0(\tilde{M}_\infty, K_{\tilde{M}_\infty}^{-1}) = 4$, we have $m \leqq 4$. It is well-known that the relatively minimal rational surface are exactly CP^2 and Hirzebruch surfaces $\sum_l (l \geqq 0)$ (cf. [BPV]). In particular, it follows that $m = 4$ and $(F_{1j}^m)^2 = 0\ (j = 1, 3)$. So $M_\infty^m = \sum_0$. Now any anticanonical divisor of $K_{\tilde{M}_\infty}^{-1}$ descends to the one of $K_{\sum_0}^{-1}$ containing F_{11}^m, F_{13}^m and $F_{11}^m \cap F_{13}^m$ with multiplicity $\geqq 3$. It follows that $h^0(M_\infty, K_{\tilde{M}_\infty}^{-1}) \leqq 3$. This contradicts to the fact that $h^0(\tilde{M}_\infty, K_{\tilde{M}_\infty}^{-1}) = 4$.

Hence x_1 can only be of type $C^2/Z_{1,3,1}$. By the same arguments as in proving Claim 1, one can easily show that $F_{11} \cdot \tilde{D}_1 = 1$, $F_{12} \cdot \tilde{D}_1 = 0$ and $F_{11}^2 = -2$. By adjunction formula, we have two exceptional curves E_1, E_2 intersecting F_{11} outside F_{12} (E_1 may coincide with E_2). In particular, it follows that any section S in $H^0(M_\infty, K_{M_\infty}^{-1})$ is locally represented by a holomorphic function $f_S \in IP(1, 3, 1)$ on $\tilde{U}_1$ with $\mathrm{ord}_o(f_S) \geqq 3$, where $\pi_1 : \tilde{U}_1 \to U_1$ is a local uniformization of M_∞ at x_1, moreover, we can write $f_S = h_{E_1} h_{E_2} \tilde{f}_S$, where $\deg_o(\tilde{f}_S) \geqq 1$ and h_{E_i} is the defining function of $\pi_1^{-1}(\pi(E_i))$ for $i = 1, 2$. Then we can choose two divisors $\tilde{D}_1$ and $\tilde{D}'_1$ in $|\tilde{D}_1|$ such that $\tilde{D}_1$ has no common component with $\tilde{D}'_1$ and $\tilde{B}_1$ and intersects with $\tilde{D}'_1$ at $8 - n$ points outside $\tilde{B}_1$, and the divisor $\pi(\tilde{D}'_1) + B_1$ defines a section S' in $H^0(M_\infty, K_{M_\infty}^{-1})$ with $\deg_o(f_{S'}) \geqq 7$. Thus by Lemma 6.1 and the fact that $6K_{M_\infty}^{-1}$ is a Cartier divisor,

$$\begin{aligned} 9 - n &= \int_{E_1} \omega_{g_\infty} + \int_{E_2} \omega_{g_\infty} + \int_{\tilde{D}_1} \omega_{g_\infty} \\ &\geqq \frac{1}{3} + \int_{D_1} \omega_{g_\infty} \end{aligned}$$

$$\geqq \frac{1}{3} + (8-n) + \frac{1}{\deg_o(\pi_1)} i_o(\pi_1^{-1}(\pi(\tilde{D}_1)), \{f_{S'} = 0\})$$

$$\geqq \frac{1}{3} + (8-n) + \frac{7}{9} > 9 - n$$

A contradiction! The lemma is proved.

Now we consider the last case that $F_1 \cdot \tilde{D}_1 = 1$, $F_2 \cdot \tilde{D}_1 = 1$ and $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 3$.

Lemma 7.3. *Let (M_∞, g_∞), F_j, $\tilde{D}_1$ be given as in Proposition 7.1. Suppose that $F_1 \cdot \tilde{D}_1 = 1$, $F_2 \cdot \tilde{D}_1 = 1$. Then M_∞ has exactly two singular points besides rational double points, one of them is of type $C^2/Z_{l,2,1}$ $(1 \leqq l \leqq 3)$ and another one is of type $C^2/Z_{1,3,1}$. Moreover,*

$$\tilde{B}_1 = F_1 + F_2 + E, \quad B_1 = \pi(E) \tag{7.8}$$

where E is an exceptional curve of first kind in minimal resolution $\tilde{M}_\infty$ of M_∞.

Proof. By adjunction formula, it can be proved that the connected component in $\tilde{B}_1$ containing F_1 is a chain of rational curves ending at F_2. Since the generic divisors in $|\tilde{D}_1|$ do not intersect $F_j (j \geqq 3)$ if such F_j exist and the anticanonical divisor in M_∞ is connected, we conclude that $\tilde{B}_1$ is a chain of rational curves with F_1 and F_2 as two ends. By Lemma 6.1, we can further conclude that $\tilde{B}_1 = F_1 + F_2 + E$ and E is an exceptional curve in $\tilde{M}_\infty$ of first kind. Also by Lemma 6.1, one can easily prove that at least one of x_1, x_2 in M_∞ must be of type $C^2/Z_{1,3,1}$. We assume that x_2 is of type $C^2/Z_{1,3,1}$.

Claim. The singular point x_1 can not be of type $C^2/Z_{1,3,1}$.

In fact, if x_1 is of type $C^2/Z_{1,3,1}$, then by Lemma 6.1, one can easily show that $\tilde{D}_1 \cdot F_{l2} = 1$, $F_{l1} \cdot F_{l2} = 1$, $F_{l1} \cdot E = 1$, $F_{l1}^2 = -2$, $F_{l2}^2 = -5 (l = 1, 2)$. Let $\tilde{M}'_\infty$ be the surface obtained by blowing down E and F_{11} in $\tilde{M}_\infty$ successively. If $\pi_1 : \tilde{M}_\infty \to \tilde{M}'_\infty$ is the natural projection, then $\pi_1(\tilde{D}_1)^2 = \tilde{D}_1^2 = 8 - n$ and $\pi_1(F_{21})^2 = 0$, $\pi_1(F_{21}) \cdot \pi_1(\tilde{D}_1) = 0$. This $\pi_1(F_{21})$ induces a fibration of $\tilde{M}'_\infty$ over CP' with generic fiber CP^1 such that $\pi_1(\tilde{D}_1)$ is contained in fibers. It contradicts to the fact that the generic $\tilde{D}_1$ is irreducible and $\pi_1(\tilde{D}_1)^2 \geqq 2$. So the claim is true.

Thus x_1 is of type $C^2/Z_{l,2,1}$, where $1 \leqq l \leqq 3$ and $l = 3$ only if $n = 6$. The lemma is proved.

Lemma 7.4. *Let (M_∞, g_∞), E, F_1, F_2, $\tilde{D}_1$ be given as in last lemma and $\tilde{B}_1 = F_1 + F_2 + E$. Then $|6K_{M_\infty}^{-1}|$ is free of base point.*

Proof. We first remark that by Lemma 6.1, the generic divisor in $|D_1|$ does not intersect B_1 outside the singular points x_1, x_2. By Lemma 7.3, we have $B_1 = \pi(E)$, where E is defined there.

Claim 1. $B_2 = \varnothing$.

We prove this claim by contradiction. Suppose that $B_2 = \varnothing$, then $B_1 \subset B_2$. Fix an irreducible divisor in $|D_1|$, say D_1 for simplicity. Then a 2-anticanonical divisor in $|2K_{M_\infty}^{-1}|$ is the sum of two anticanonical divisors if it contains D_1. On the other hand, by Lemma 5.2, we have $h^0(M_\infty, K_{M_\infty}^{-1}) = 10 - n$,

$h^0(M_\infty, 2K_{M_\infty}^{-1}) = 28 - 3n$, where $n = 5$ or 6. Thus there is a global section S_2^∞ of $H^0(M_\infty, 2K_{M_\infty}^{-1})$ such that $\{S_2^\infty = 0\}$ does not contain D_1 and intersects with D_1 at $17 - 2n$ points outside B_1. By Lemma 7.3, we may assume that x_1 is the singular point of type $C^2/Z_{l,2,1}$ $(1 \leqq l \leqq 3)$. Let $\pi: \tilde{U}_{x_1} \to U_{x_1}$ be the local uniformization of M_∞ with $\pi_1(0) = x_1$ and f_2 be the holomorphic function locally representing $\pi_1^* S_2^\infty$ in $\tilde{U}_{x_1}$. Then f_2 is invariant under the action of $Z_{l,2,1}$, i.e., $\sigma^* f_2 = f_2$, where $\sigma \in Z_{l,2,1}$ is the generator. Then we can choose a local coordinate system (z_1, z_2) on $\tilde{U}_{x_1}$ such that $\pi_1^{-1}(D_1) = \{z_1 = 0\}$ and $\sigma = \sigma_{l,2,1}$ as defined in Lemma 5.5. It follows from σ-invariance of f_2 and f_2 is a holomorphic function on $z_1^{4l}, z_2^{4l}, (z_1 z_2)^2$, $z_1^{2l+1} z_2$, $z_1 z_2^{2l+1}$. Then one can easily deduce

$$i_0(\{f_2 = 0\}, \pi_1^{-1}(D_1)) \geqq 4l \tag{7.9}$$

By Lemma 6.1 and the above (7.9), we have

$$\begin{aligned} 18 - 2n &= 2\int_{B_1} \omega_{g_\infty} + 2\int_{D_1} \omega_{g_\infty} \\ &\geqq \frac{1}{3} + 17 - 2n + \frac{1}{4l} i_0(\{f_2 = 0\}, \pi_1^{-1}(D_1)) \\ &\geqq \frac{1}{3} + 18 - 2n \end{aligned} \tag{7.10}$$

A contradiction! The claim is proved.

The above claim implies that the base locus of $|2K_{M_\infty}^{-1}|$ consists of finitely many points. Define

$$\psi = 3\log\left(\sum_{\beta=0}^{N_2} \|S_{2\beta}^\infty\|_{g_\infty}^2\right) \tag{7.11}$$

where $N_2 = h^0(M_\infty, 2K_{M_\infty}^{-1}) - 1$ and $\{S_{2\beta}^\infty\}$ is an orthonormal basis of $H^0(M_\infty, 2K_{M_\infty}^{-1})$ with respect to g_∞. Then ψ is smooth outside the base locus of $|2K_{M_\infty}^{-1}|$. The rest of the proof is exactly same as that from (7.2) to the end in the proof of Lemma 7.1.

Now Proposition 7.2 follows from Lemma 7.2, 7.2 and 7.4. Then the proof of Theorem 2.2 is completed.

The discussions in previous sections also yield the following result on the degeneration of Kähler-Einstein surfaces with positive scalar curvature.

Theorem 7.1. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein surfaces with $C_1(M_i)^2 = 9 - n$ $(5 \leqq n \leqq 8)$. Then by taking the subsequence, we may have that $(M - i, g_i)$ converge to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Proposition 4.2 satisfying:*

(1) *if $n = 8$, then M_∞ has at most one singular point of type $C^2/Z_{l,2,1}$ $(2 \leqq l \leqq 7)$ besides rational double points;*
(2) *if $n = 7$, then M_∞ has either only rational double points or two singular points of type $C^2/Z_{1,2,1}$ besides rational double points;*
(3) *if $n = 5, 6$, M_∞ has at most two singular points of type $C^2/Z_{l,2,1}$ or $C^2/Z_{1,3,1}$*

($1 \leqq l \leqq 3$) *besides rational double points, moreover, in case* M_∞ *has two of such singular points, one of them must of type* $C^2/Z_{l,2,1}$, *while another one is of type* $C^2/Z_{1,3,1}$.

This theorem generalizes some results of M. Anderson [An] and Nakajima [Na] on the Hausdorff convergence of Einstein 4-manifolds with positive scalar curvature in case of complex geometry. Precisely, this theorem gives the reduction of quotient singular points in the limit Einstein orbifold which is the Hausdorff limit of a sequence of Kähler-Einstein surfaces. As we have already seen, such a reduction is, in general, completely nontrivial. In fact, we expect that M_∞ in Theorem 7.1 has only rational double points as singular points. If it is true, then we have stronger partial C^0-estimate than that in Theorem 2.2 and can simplify a lot of technical computations in section 2 and Appendices.

Appendix 1. Proof of Lemma 2.4

In this appendix, we will prove Lemma 2.4 stated in section 2. First we will prove a proposition concerning the evaluation of rational integrals.

Let f be a holomorphic function defined in the ball $B_R(o) \subset C^2$ with center at the origin o. For simplicity, take $R = 1$. For any $\varepsilon \in (0, \frac{1}{2})$, $x \in B_{\frac{1}{2}}(o)$ and $\alpha > 0$, we write

$$I_\varepsilon(f, \alpha, x) = \int_{B_\varepsilon(x)} \frac{dV}{|f|^{2\alpha}}$$

where dV denotes the standard euclidean volume form on C^2.

Then we can associate a local analytic invariant $\alpha_x(f)$ to f at any point x in $B_{\frac{1}{2}}(o)$ as follows,

$$\alpha_x(f) = \sup\{\alpha | \exists \varepsilon > 0, \text{ s.t. } I_\varepsilon(f, \alpha, x) < \infty\} \tag{A.1.1}$$

Note that $\alpha_x(f)$ is independent of choices of local holomorphic coordinates at x.

We would like to evaluate $\alpha_x(f)$ in terms of the geometry of Z_f at x, where Z_f is the zero locus of f in $B_1(o)$. Obviously, if $x \notin Z_f$, then $\alpha_x(f) = +\infty$. Furthermore, by some elementary computations, one can easily check that if x is the smooth point of the reduced curve $(Z_f)_{\text{red}}$ of Z_f in $B_{\frac{1}{2}}(o)$ and m is the multiplicity of Z_f at x, then $\alpha_x(f) = \dfrac{1}{m}$. Note that $(Z_f)_{\text{red}}$ is defined as follows, write $Z_f = \alpha_1 Z_1 + \ldots + \alpha_k Z_k$, where $Z_i (1 \leqq i \leqq k)$ are distinct irreducible components of Z_f in $B_{\frac{1}{2}}(o)$, then $(Z_f)_{\text{red}} = Z_1 + \ldots + Z_k$. Therefore, it suffices to evaluate $\alpha_x(f)$ at the singular point of $(Z_f)_{\text{red}}$ in $B_{\frac{1}{2}}(o)$. Without losing generality, we may assume that o is the unique singular point of $(Z_f)_{\text{red}}$ in $B_{\frac{1}{2}}(o)$.

Given any local coordinates (z_1, z_2) of C^2 at o, one can expand f in a power series $\sum_{i,j \geqq 0} a_{ij} z_1^i z_2^j$ in a small neighborhood of o. Then we define the Newton polyhedron $N(f)$ of f as the convex hull of the set $\{(i, j) \in R_+ \times R_+, (0, +\infty), (+\infty, 1) | a_{ij} \neq 0\}$ in R^2. The boundary $\partial N(f)$ intersects with the line $\{x = y\}$ in R^2 at a point (x_f, y_f), where x, y are euclidean coordinates of R^2, $x_f = y_f$. This x_f is called in [AGV] the remotedness of $N(f)$, denoted by $r(N(f))$. Since $N(f)$ obviously depends on the choice of local coordinates C^2 at o, so does $r(N(f))$. However, we have

Proposition A.1.1. *Let* f *be given as above. Then there are a sequence of coordinate system* (z_1^m, z_2^m) *of* C^2 *at* o *such that for the associated Newton polyhedron* $N_m(f)$ *in the coordinate system* (z_1^m, z_2^m), *we have* $r(N_m(f)) \leqq r(N_{m'}(f))$ *if* $m \leqq m'$ *and* $\alpha_o(f) = \lim_{m \to \infty} r(N_m(f))^{-1}$.

Proof. It is clear that $\alpha_o(f) \leqq \dfrac{1}{m_o}$. Write $f = f_k + f_{k+1} + \ldots$, where each f_j is the homogeneous component of f with degree j, $k = \text{mult}_o(f)$. Obviously, $k_0 \geqq m_0$. Take $\alpha < m_0^{-1}$, then for any

$\varepsilon > 0$,

$$\int_{B_{\frac{1}{2}}(o)\setminus B_\varepsilon(o)} \frac{dV}{|f|^{2\alpha}} < +\infty \tag{A.1.2}$$

By a linear transformation, we may assume that $f_k = z_1^a z_2^b \prod_{i=1}^{l} (z_1 + \mu_i z_2)^{c_i}$ with $\mu_i \neq 0$ distinct, $a \geqq b \geqq c_i (i = 1, 2, \ldots, l)$ and $a + b + \sum_{i=1}^{l} c_i = k$. Let $N(f)$ be the associated Newton polyhedron in R^2 of f in this coordinate system.

Case 1. $\partial N(f) \cap \{x = y\} \cap \{x + y = k\} \neq \varnothing$. Note that in this proof we always use x, y to denote the coordinates of R^2, z_1, z_2 to denote the coordinates of C^2.

In this case, we will prove that $\alpha_0(f) = 2/k$. In fact, for $\delta > 0$ sufficiently small, we have

$$I_\delta(f, \alpha, o) = \int_{B_\delta(o)} \frac{dV}{|f|^{2\alpha}} = \int_{\substack{|x| \leqq |y| \\ |x|^2 + |y|^2 \leqq \delta}} \frac{dV}{|f|^{2\alpha}} + \int_{\substack{|y| \leqq |x| \\ |x|^2 + |y|^2 \leqq \delta}} \frac{dV}{|f|^{2\alpha}} \tag{A.1.3}$$

$$\leqq \int_{\substack{|y| \leqq \delta \\ |\xi| \leqq 1}} \frac{dy \wedge d\bar{y} \wedge d\xi \wedge d\bar{\xi}}{|y|^{2k\alpha - 2} \left| \xi^a \prod^{l} (\xi + \mu_i)^{c_i} + \sum_{m \geqq k+1} y^{-k} f_m(y\xi, y) \right|^{2\alpha}}$$

$$+ \int_{\substack{|x| \leqq \delta \\ |\eta| \leqq 1}} \frac{dx \wedge d\bar{x} \wedge d\eta \wedge d\bar{\eta}}{|x|^{2k\alpha - 2} \left| \eta^b \prod_{i=1}^{l} (1 + \mu_i \eta)^{c_i} + \sum_{m \geqq k+1} x^{-k} f_m(x, x\eta) \right|^{2\alpha}}$$

It follows that $I_\delta(f, \alpha, o) < +\infty$ only if $\alpha < \frac{2}{k}$. On the other hand, by our assumption, $b + \sum_{i=1}^{l} c_i \geqq a$, $a \geqq b \geqq c_i$, so $\max\{c_i, b, a\} \leqq \frac{k}{2}$. Thus both polynomials $\xi^a \prod_{i=1}^{l} (\xi + \mu_i)^{c_i}$ and $\eta^b \prod_{i=1}^{l} (1 + \mu_i \eta)^{c_i}$ have roots with multiplicity $\leqq \frac{k}{2}$. We also have

$$\sum_{m \geqq k+1} y^{-k} f_m(y\xi, y) = 0 \quad \text{at } y = 0$$

$$\sum_{m \geqq k+1} x^{-k} f_m(x, x\eta) = 0 \quad \text{at } x = 0$$

Therefore, for δ sufficiently small, both integrals in (A.1.3) are finite if $\alpha < \frac{2}{k}$. So $\alpha_o(f) = \frac{2}{k}$. Actually, we have already proved that $I_\delta(f, \alpha, o)$ has a upper bound depending on δ, α and the upper bound of $|f|$ in $B_{\frac{1}{2}}(o)$.

Case 2. $\partial N(f) \cap \{x = y\} \cap \{x + y = k\} = \varnothing$.

Now $a > b + \sum_{i=1}^{l} c_i$. Let $L(u, v) = \{ux + vy = 1\}$ be the unique line containing the segment of $\partial N(f)$ having nonempty intersection with $\{x = y\}$ in R^2. Thus $\alpha_o(f) = r(N(f))^{-1}$. In the following, we may assume that $L(u, v)$ is not vertical. In fact, if $L(u, v)$ is vertical, then by Fubini theorem, one can easily check that $I_\varepsilon(f, \alpha, o)$ is finite iff $\alpha < r(N(f))^{-1}$. Note that $r(N(f))$ is the distance of the line $L(u, v)$ from y-axis in R^2. Let (i, j) and (i', j') be the two end points of $L(u, v) \cap \partial N(f)$ with $i > j$, $i' < j'$. We further have that $i > i'$, $j < j'$ and $i' + j' \geqq i + j$. A simple computation shows

$$u = \frac{j' - j}{ij' - ji'}, \quad v = \frac{i - i'}{ij' - ji'}$$

Then there are integers $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ such that $u = \frac{\tilde{\alpha}}{\tilde{\gamma}}$, $v = \frac{\tilde{\beta}}{\tilde{\gamma}}$ and $\tilde{\alpha}$, $\tilde{\beta}$ are coprime.

Define a polynomial f_L by $\sum_{(k,l)\in L(u,v)\cap\partial N(f)} a_{kl} z_1^k z_2^l$. There are at most i terms in f_L. This polynomial f_L has the decomposition,

$$f_L(z_1, z_2) = c z_1^{i'} z_2^{j} \prod_{k=1}^{v} (z_1^{\tilde\beta} + d_k z_2^{\tilde\alpha})^{\mu_k} \tag{A.1.4}$$

where $c \neq 0$ and $d_1, \ldots, d_v$ are distinct constants. Note that $\sum_{k=1}^{v} \tilde\beta\mu_k + i' = i$, $\sum_{k=1}^{v} \tilde\alpha\mu_k + j = j'$. For $\delta > 0$ sufficiently small, let $\delta_1 = \delta^{\tilde\alpha}$, $\delta_2 = \delta^{\tilde\beta}$, $\delta' = \min\{\delta_1, \delta_2\}$, then we have

$$I_{\delta'}(f, \alpha, o) \leqq \int_{\substack{|z_1|\leqq\delta_1\\|z_2|\leqq\delta_2}} \frac{dV}{|f(z_1,z_2)|^{2\alpha}} = \tilde\alpha\tilde\beta \int_{\substack{|w_1|\leqq\delta\\|w_2|\leqq\delta}} \frac{|w_1|^{2\tilde\alpha-2}|w_2|^{2\tilde\beta-2}\, dw_1\wedge d\bar w_1\wedge dw_2\wedge d\bar w_2}{|f(w_1^{\tilde\alpha}, w_2^{\tilde\beta})|^{2\alpha}} \tag{A.1.5}$$

$$= \tilde\alpha \int_{\substack{|w_1|\leqq\delta\\|\eta|\leqq 1}} \frac{dw_1\wedge d\bar w_1\wedge d\eta\wedge d\bar\eta}{|w_1|^{2\alpha\tilde\gamma - 2\tilde\alpha - 2\tilde\beta + 2}\left|c\eta^j\prod_{k=1}^{v}(1+d_k\eta)^{\mu_k} + w_1^{-\tilde\gamma}\tilde f(w_1^{\tilde\alpha}, w_1^{\tilde\beta}\eta)\right|^{2\alpha}}$$

$$+ \tilde\beta \int_{\substack{|w_2|\leqq\delta\\|\xi|\leqq 1}} \frac{dw_2\wedge d\bar w_2\wedge d\xi\wedge d\bar\xi}{|w_2|^{2\alpha\tilde\gamma - 2\tilde\alpha - 2\tilde\beta + 2}\left|c\xi^{i'}\prod_{k=1}^{v}(\xi+d_k)^{\mu_k} + w_1^{-\tilde\gamma}\tilde f(w_2^{\tilde\alpha}\xi, w_2^{\tilde\beta})\right|^{2\alpha}}$$

where $\tilde f = f - f_L$. It follows immediately from (A.1.5) that $I_{\delta'}(f, \alpha, 0) < +\infty$ only if $\alpha < \dfrac{\tilde\alpha+\tilde\beta}{\tilde\gamma}$. By the definition of $r(N(f))$, we have $i', j \leqq r(N(f)) = \dfrac{\tilde\gamma}{\tilde\alpha+\tilde\beta}$. Since $w_1^{-\tilde\gamma}\tilde f(w_1^{\tilde\alpha}, w_1^{\tilde\beta}\eta) = 0$ at $w_1 = 0$ and $w_2^{-\tilde\gamma}\tilde f(w_2^{\tilde\alpha}\xi, w_2^{\tilde\beta}) =$ at $w_2 = 0$, one can easily show that $I_{\delta'}(\alpha, f, 0) < \infty$ if $\max_{1\leqq k\leqq v}\{\mu_k\} < \dfrac{1}{\alpha}$ and $\alpha < r(N(f))^{-1}$. Thus if $\mu_k \leqq r(N(f))$ for all k, then $I_{\delta'}(\alpha, f, 0) < \infty$ iff $\alpha < (r(N(f)))^{-1}$, i.e., $\alpha_o(f) = r(N(f))^{-1}$ and the proposition is proved. Otherwise, there is a $\mu_k > r(N(f))$. For simplicity, say $k = 1$, i.e., $\mu_1 > r(N(f))$. Then by the fact that $\dfrac{\tilde\gamma}{\tilde\alpha+\tilde\beta} > \dfrac{i}{2}$, we derive $\tilde\beta = 1$, $\tilde\alpha \geqq 2$. Note that other μ_k for $k \geqq 2$ are all less than $r(N(f))$.

Making a transformation $(z_1, z_2) \to (z_1 + \mu_1 z_2^{\tilde\alpha}, z_2)$, we obtain a new local holomorphic function g at o with f_k as the first homogeneous term g_k. Also we observe that all points on $\partial N(f)$ below (i, j) are unchanged and are still the ones on $\partial N(g)$ below $(i, j) \in \partial N(g)$. Put $N_0(f) = N(f)$, $N_1(f) = N(g)$. Note that g is just the function f in the new coordinate system. The above process can be carried out successively to obtain Newton polyhedrons $N_0(f), N_1(f), \ldots$, unless for some m, $\alpha_0(f) = r(N_m(f))^{-1}$ and the proposition is proved. Suppose that such a m does not exists, then we have a sequence of Newton polyhedrons $N_0(f), N_1(f), \ldots$ in R^2. Moreover, the parts of $\partial N_m(f)$ $(0 \leqq m < \infty)$ below the line $\{x = y\}$ are all same. Then one can easily check that $\lim_{m\to\infty} r(N_m(f)) = i$. By previous discussions, $I_\delta(f, \alpha, 0) < +\infty$ for sufficiently small $\delta > 0$ if $\alpha < i^{-1}$, and $\alpha_o(f) \leqq r(N_m(f))^{-1}$ for all m. Thus $\alpha_o(f) = i^{-1}$, the proposition is proved.

Lemma A.1.2. *Let M be a smooth complete intersection of two quadratic polynomials in CP^4 and S be a global section in $H^0(M, K_M^{-6})$ such that its zero locus $Z(S)$ contains no curve with multiplicity 9 and $Z(S)_{\text{red}}$ is not a union of two lines and a curve of degree 2 intersecting at one point. Then there is an $\varepsilon > 0$ such that for any $\alpha \leqq \frac{1}{9} + \varepsilon$,*

$$\int_M \frac{dV_{\tilde g}}{\|S\|_{\tilde g}^{2\alpha}} < +\infty \tag{A.1.6}$$

where $\tilde g$ is any fixed Kähler metric on M.

Proof. By our assumption, there are finitely many points $x_1, \ldots, x_l \in M$ such that for any $\delta > 0$,

$\alpha < \frac{3}{4}$ (cf. Fact(†) in the proof of Lemma 2.3),

$$\int_{M\setminus\bigcup_{i=1}^{l} B_\delta(x_i,\tilde{g})} \frac{dV_{\tilde{g}}}{\|S\|_{\tilde{g}}^{2/3}} < +\infty \tag{A.1.7}$$

where $B_\delta(x_i, \tilde{g})$ is the geodesic ball at x_i with respect to $\tilde{g}$. At each point x_i, the section S is locally represented by a holomorphic function f_i, we define $\alpha_{x_i}(S) = \alpha_{x_i}(f_i)$. This $\alpha_{x_i}(S)$ is in fact independent of the choice of local representations of S. Then one can easily see that the lemma is equivalent to $\alpha_{x_i}(S) > \frac{1}{6} + 2\varepsilon$ for $1 \leqq i \leqq l$, Suppose that the lemma is false, then there is a x_i, say x_1, such that $\alpha_{x_i}(S) \leqq \frac{1}{6}$. We will derive a contradiction to our assumption on S.

Let (z_1, z_2) be the local holomorphic coordinates of M at x_1, and S be represented by a local holomorphic function f_S, then

$$\mathrm{mult}_o(f_S) \geqq \alpha_o(f_S)^{-1} = \alpha_{x_1}(S)^{-1} = 9\,. \tag{A.1.8}$$

Claim 1. There are at most one line of M through the point x_1.

We prove it by contradiction. Suppose that there are two lines L_1, L_2 of M through x_1, then there is a unique anticanonical divisor $D = L_1 + L_2 + E$, here E is a curve of degree 2 with respect to K_M^{-1} and $x_1 \in E$. Note that by smoothness of M, L_1, L_2, E must intersect to each other transversally at x_1. By $K_M^{-1} \cdot L_i = 6$ for $i = 1, 2$, we have that $2L_1 + 2L_2 \subset Z(S)$. Let l_1, l_2 be local defining functions of L_1, L_2, then $f_S = l_1^2 l_2^2 h$. By Hölder inequality, the facts that $\alpha_{x_1}(S) \leqq \frac{1}{6}$ and $\alpha_0(l_1 l_2) = 1$, we derive $\mathrm{mult}_o(h) \geqq 7$. Thus $8 = (K_M^{-6} - 2L_i)\cdot L_i \geqq 9$ unless $3L_i \subset Z(S)$ for $i = 1, 2$. Hence, $3L_1 + 3L_2 \subset Z(S)$, i.e., $f_S = (l_1 l_2)^3 h_1$. Now $\mathrm{mult}_o(h) \geqq 6$. We claim that $\mathrm{mult}_0(h) \geqq 7$. In fact, if not, then by Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{6}$, for any $p > 0$, there is a local coordinate system (z_1, z_2) such that $h_1(z_1, z_2) = \sum_{i,j \geqq 0} a_{ij} z_1^i z_2^j$ at $(0, 0)$ and $a_{ij} = 0$ if either $\frac{i}{6} + \frac{j}{p} < 1$ or $\frac{i}{6} + \frac{j}{p} = 1$, $j \geqq 1$. In particular, the lowest homogeneous term f_{12} of f_S is of form $l_1^3 l_2^3 z_1^6$. Thus by Proposition A.1.1, $\alpha_0(f) \geqq \frac{1}{6}$ unless one of L_1, L_2 is tangent to $\{z_1 = 0\}$ at x_1. Assume that L_1 does so. If $4L_1 \not\subset Z(S)$, then

$$\begin{aligned} 9 &= (6K_M^{-1} - 3L_1)\cdot L_1 = 3 + \{h_1 = 0\}\cdot L_1 \\ &\geqq 3 + \inf\{2i + j \mid a_{ij} \neq 0\} \\ &\geqq 3 + 13 = 16 \end{aligned}$$

A contradiction! Therefore, $4L_1 \subset Z(S)$. One can actually prove that $5L_1 \subset Z(S)$. Choose local coordinates (z_1, z_2) such that $L_i = \{z_i = 0\}$ for $i = 1, 2$. By Proposition A.1.1 and $\alpha_o(S) \leqq \frac{1}{6}$, if we write $f_S = z_1^k z_2^3 h_2(z_1, z_2)$ where $8 \geqq k \geqq 5$, then $h_2(z_1, z_2) = (z_1 + z_2^\beta)^{9-k} + \tilde{h}_2(z_1, z_2)$ and $\tilde{h}_2$ does not have terms $z_1^i z_2^j$ with $i + j/\beta \leqq 9 - k$. By some direct computations, we can obtain $\alpha_o(S) > \frac{1}{6}$, a contradiction! Thus we must have that $\mathrm{mult}_o(h_1) \geqq 7$, so $\mathrm{mult}_o(f_S) \geqq 13$. Since $K_M^{-6}\cdot E = 12$, we conclude that $E \subset Z(S)$, so $S = S'\cdot S_5$, where S_5 is a global section of K_M^{-5}. By Hölder inequality, $\alpha_{x_1}(S_5) \leqq \frac{2}{13}$. Repeat the above arguments for S_5, we can conclude that $S_5 = S'\cdot S_4$ with $\alpha_{x_1}(S_4) \leqq \frac{1}{6}$. Inductively, we finally obtain $S = (S')^6$. By the definition of S', it contradicts to our assumption on S. Therefore, Claim 1 is proved.

Claim 2. There is no line of M through x_1.

If not, by Claim 1, there is exactly one line L_1 of M with $x_1 \in L_1$. As before, $2L_1 \subset Z(S)$. By the above arguments in the proof of Claim 1, it follows from $\alpha_{x_1} \leqq \frac{1}{6}$ that $\mathrm{mult}_o(f_S) \geqq 10$, where f_S is the local holomorphic representation of S at x_1.

Let f_k be the lowest homogeneous term of f_S at x_1. We may assume that there are k_1, k_2 with $k_1 + k_2 = k$ satisfying

$$k_1 = \max\{l_1, l_2 \mid l_1 + l_2 = k,\ z_1^{l_1} z_2^{l_2} \text{ is in } f_k\} \tag{A.1.9}$$

In particular, $k_1 \geqq k_2$. Give a partial order on monomials $z_1^{l_1} z_2^{l_2} : z_1^{l_1} z_2^{l_2} \leqq z_1^{l_1'} z_2^{l_2'}$ if $l_1 \leqq l_1'$. Let $z_1^{j_1} z_2^{j_2}$ be the smallest term in f_k with respect to the above partial order. If $j_1 \leqq j_2$, then by the proof of Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{6}$, $k \geqq 18$. Choose an anticanonical divisor S' such that $Z(S') = L_1 + D$ and D is tangent to L_1 at x_1. By Claim 1 and the properties of M, the divisor D is

irreducible and smooth at x_1. If $Z(S)$ does not contain D, then

$$18 = Z(S)\cdot D \geqq 16 + 2L_1\cdot D = 20$$

A contradiction! So $Z(S')\subset Z(S)$. Inductively, we can prove $S=(S')^6$. But $\mathrm{mult}_{x_1}((S')^6)=12$. It contradicts to that $k\geqq 18$. Thus $j_1>j_2$. In particular, $k_1>k_2$. Note that the same arguments as above show that $k\leqq 14$. By the geometric properties of M, one can easily check that there are exactly five curves $D_i (1\leqq i\leqq 5)$ of degree 2 through the point x_1 and with distinct tangent directions at x_1. If $k=14$, then $Z(S)=4L_1+2\sum_{i=1}^5 D_1$. It implies that $\alpha_{x_1}(S)\geqq \frac{1}{7}$, impossible! If $k=13$, then $4L_1+\sum_{i=1}^5 D_1\subset Z(S)$. One can also show that it will be against our assumption $\alpha_{x_1}(S)\leqq\frac{1}{5}$. Thus $k\leqq 12$. Since $\alpha_{x_1}(S)\leqq\frac{1}{5}$, $j_1\geqq 9$ and $j_2\leqq 3$, where $z_1^{j_1}z_2^{j_2}$ is the smallest term in f_k.

Choose an anticanonical section S'' such that $Z(S'')=L_1+D'$, where D' is tangent to $\{z_1=0\}$ at x_1.

By Proposition A.1.1 and $\alpha_{x_1}(S)\leqq\frac{1}{5}$, we can choose (z_1,z_2) such that the Taylor expansion of f_S at x_1 does not have terms $z_1^i z_2^j$ with $i(9-j_2-\delta)+j(j_1-9+\delta)<(9-\delta)(j_1-j_2)$, where $\delta>0$ is sufficiently small. Then if $D'\not\subset Z(S)$, we have

$$\begin{aligned} Z(S)\cdot D' &\geqq \min\{2i+j \mid z_1^i z_2^j \text{ is in } f_S\} \\ &\geqq \min\{2i+j \mid z_1^i z_2^j \text{ is in } f_S,\ i\leqq 8\} \\ &\geqq 18+\frac{27-2j_1-j_2}{j_1-9}(9-i)-\delta', \end{aligned} \tag{A.1.10}$$

where δ' is small and depends on δ. First we assume that D' is irreducible, then by (A.1.10), $D'\subset Z(S)$, so $S=S''\cdot S_5$, where S_5 is a global section of K_M^{-5}. One can compute that

$$\int_M \frac{dV_{\tilde g}}{\|S''\|_{\tilde g}^{2\beta}}<\infty \quad \text{if } \beta<\frac{3}{4}. \tag{A.1.11}$$

Then by $\alpha_{x_1}(S)\leqq\frac{1}{5}$, we have $\alpha_{x_1}(S_5)\leqq\frac{3}{23}$. In fact, one can easily prove that L_1 is also tangent to $\{z_1=0\}$. Inductively, we will conclude that $S=(S'')^6$. It implies that $\alpha_{x_1}(S)\geqq\frac{1}{6}\alpha_{x_1}(S'')=\frac{1}{8}$. A contradiction! Therefore, D' is reducible. It will have distinct tangent direction from that of L_1 at x_1. Then we have that $j_1=9$, $j_2=3$ and $L_1=\{z_2=0\}$. By $K_M^{-6}\cdot D'=12$, Proposition A.1.1 and D' is tangent to $\{z_1=0\}$, one can deduce that $5D'+2L_1\subset Z(S)$. Then by the arguments in the proof of Claim 1, we will have either $\alpha_{x_1}(S)>\frac{1}{5}$ or $9D'\subset Z(S)$. Since both cases are impossible, we complete the proof of Claim 2.

From now on, we may assume that no line of M passes through x_1. Then there is a pencil of anticanonical divisors such that the generic one of it is irreducible and vanishes at x_1 of order 2. In particular, it implies that $k\leqq 12$.

By Proposition A.1.1 and $\alpha_{x_1}(S)\leqq\frac{1}{5}$, we can choose local coordinates (z_1,z_2) such that the Taylor expansion of f_S at x_1 does not have terms $z_1^i z_2^j$ with $(9-\delta-j_1)i+(j_1-9+\delta)j\leqq(9-\delta)(j_1-j_2)$, where j_1,j_2 are given in the proof of Claim 2, δ is sufficiently small number. On the other hand, we have an anticanonical section S' such that its local holomorphic representation $h_{S'}$ is of form $z_1\tilde h_1(z_1,z_2)+z_2^3\tilde h_2(z_2)$ at x_1.

Case 1. $Z(S')$ is irreducible.

If $Z(S)$ does not contain $Z(S')$, then

$$\begin{aligned} 24 &= Z(S)\cdot Z(S') \\ &\geqq \min\{2i+j \mid z_1^i z_2^j \text{ is in } f_S\}+\min\{i+j \mid z_1^i z_2^j \text{ is in } f_S\} \\ &\geqq 18+9=27 \end{aligned}$$

A contradiction! Thus $S=S'\cdot S_5$. An easy computation shows that $\alpha_{x_1}\geqq\frac{1}{4}$, so $\alpha_{x_1}(S_5)\leqq\frac{3}{23}$.

Case 2. $Z(S')$ is reducible.

By Claim 2, the divisor $Z(S')$ consists of two curves D_1, D_2 of degree 2 such that both D_1 and D_2 are smooth at x_1, D_1 is tangent to $\{z_1=0\}$. As in Case 1, we have $D_1\subset Z(S)$. Let g_1 be the defining equation of D_1 at x_1, then $\alpha_0(g_1)=1$. So if we decompose $f_S=g_1\tilde f$, then $\alpha_0(\tilde f)\leqq\frac{1}{8}$,

$\text{mult}_o(f) \leqq 11$. Then one can deduce $2D_1 \subset Z(S)$. In fact, the same arguments show that $3D_1 \subset Z(S)$. On the other hand, arguments in the proof of Claim 1, one can show that $k = \text{mult}_o(f_S) \geqq 10$. Thus if $D_2 \not\subset Z(S)$, we would have

$$12 = D_2 \cdot Z(S) = 3D_2 \cdot D_1 + D_2(Z(S) - 3D_1))$$
$$\geqq 6 + 7 = 13$$

A contradiction! Therefore $D_2 \subset Z(S)$, i.e., $S = S'S_5$. Also $\alpha_{x_1}(S) \leqq \frac{3}{23}$.

Inductively, we can prove that $S = (S')^6$, then $\alpha_{x_1}(S) \geqq \frac{1}{8}$. It contradicts to our assumption. Thus the lemma is proved.

Lemma A.1.2. *Let M be a smooth cubic surface in CP^3 and S be a section of K_M^{-6}. Assume that $Z(S)_{\text{red}}$ is not an anticanonical divisor consisting of three lines intersecting at a common point. Then there is an $\varepsilon > 0$ such that for $\alpha \leqq \frac{2}{3} + \varepsilon$,*

$$\int_M \frac{dV_{\tilde{g}}}{\|S\|_{\tilde{g}}^{\alpha/3}} < \infty \tag{A.1.12}$$

where $\tilde{g}$ is any given Kähler metric on M.

Proof. As in the proof of last lemma, it suffices to prove that $\alpha_{x_i}(S) > \frac{1}{6}$ for $1 \leqq i \leqq l$, while $\alpha_x(S) > \frac{1}{6}$ for any $x \neq x_i$ $(1 \leqq i \leqq l)$. Assume that $\alpha_{x_1}(S) \leqq \frac{1}{6}$, we will derive a contradiction. Since the proof is identical to that of last lemma, we just sketch it. Note that there are at most two lines through x_1.

Case 1. There are exactly two lines L_1, L_2 of M through x_1.

In this case, there is an anticanonical section S' of K_M^{-1} with $Z(S') = L_1 + L_2 + L_3$, where L_3 is a line of M not through x_1. Then as in the proof of Claim 1 in Lemma A.1.1, either $4L_1 + 3L_2$ or $3L_1 + 4L_2$ is in $Z(S)$. But $L_1 \cdot L_3 = 1$, $L_2 \cdot L_3 = 1$, so $L_3 \subset Z(S)$, i.e., $S = S' \cdot S_5$. One can directly compute that $\alpha_{x_1}(S') = 1$, so $\alpha_{x_1}(S_5) \leqq \frac{1}{8}$. Inductively, one can show that $S = (S')^6$, then $\alpha_{x_1}(S) \geqq \frac{1}{6}$. A contradiction.

Case 2. There is exactly one line L_1 of M through x_1.

Let S' be the section of K_M^{-1} with $Z(S') = L_1 + D$, where D is an irreducible curve of degree 2 containing x_1. As before, $2L_1 \subset Z(S)$. Let f_S be the local holomorphic representation of S in some local coordinates (z_1, z_2), and l_1 be the defining equation of L_1, then $f_S = l_1^2 h$, $\text{mult}_o(h_1) \geqq 7$ and $\alpha_o(h_1) \leqq \frac{1}{7}$. Let f_k, h_{k-2} be the lowest homogeneous terms of f_S, h, respectively. Then we may assume that $h_{k-2} = z_1^{j_1} z_2^{j_2} + \ldots$ and any term $z_1^i z_2^j$ in h_{k-2} has the property: $i \geqq j_1$. If L_1 is not tangent to $\{z_1 = 0\}$, then $f_k = \lambda z_1^{j_1} z_2^{j_2+2} + \ldots$ satisfying: $\lambda \neq 0$ and no $z_1^i z_2^j$ in f_k with $i < j_1$. By the proof of Proposition A.1.1, $j_1 \geqq 9$, so $k \geqq 11$ and $3L_1 \subset Z(S)$. Consequently, it follows from $L_1 \cdot D = 2$ that $D \subset Z(S)$, i.e., $S = S' \cdot S_5$. If L_1 is indeed tangent to $\{z_1 = 0\}$, then we can prove that $4L_1 \subset Z(S)$, so $D \subset Z(S)$, otherwise,

$$12 = D \cdot Z(S) \geqq 4L_1 \cdot D + (k - 4) \geqq 8 + 5 = 13$$

A contradiction!

Therefore, we always have $S = S' \cdot S_5$. Then inductively, one can actually prove that $S = (S')^6$, so $\alpha_{x_1}(S) = \frac{1}{6}\alpha_{x_1}(S') \geqq \frac{1}{8}$. A contradiction.

Case 3. There is no line of M through x_1.

Let S' be the anticanonical section vanishing at x_1 of order 2. Then $Z(S')$ is irreducible. If $Z(S') \subset Z(S)$, then $S = S' \cdot S_5$, where S_5 is a global section of K_M^{-5}. One can directly check that $\alpha_{x_1}(S') = \frac{5}{6}$, so $\alpha_{x_1}(S_5) \leqq \frac{1}{39}$. In particular, S_5 vanishes at x_1 of order at least 8. Thus by $Z(S_5) \cdot Z(S') = 15$, we conclude $Z(S') \subset Z(S_5)$. Inductively, we have $S = (S')^6$, so $\alpha_{x_1}(S') = \frac{1}{6}\alpha_{x_1}(S') > \frac{1}{6}$. A contradiction.

Now $Z(S') \not\subset Z(S)$. It implies that $k = \text{mult}_o(f_S) = 9$. By Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{6}$, we can choose local coordinates (z_1, z_2) such that f_S does not have terms $z_1^i z_2^j$ in its power series at x_1 if $\frac{i}{9} + \frac{j}{p} < 1$, where p is a sufficiently large integer. In particular, the lowest homogeneous term f_k of f_S is z_1^9. By using the holomorphic transformation of form $(z_1, z_2) \to (z_1 + \sum_{k \geqq 1} b_k z_2^k, z_2)$, we can

eliminate the monomials $z_1^k z_2^l$ for $l \geqq 1$ in the Taylor expansion of f_S at o. Then the proof of proposition A.1.1 implies that either $\alpha_{\tilde{S}_1}(S) > \frac{1}{3}$ or f_S contains a curve with multiplicity 9. Both cases are impossible! Thus we also derive a contradiction in Case 3. The lemma is proved.

The Lemma 2.4 in section 2 obviously follows from Lemma A.1.1 and A.1.2.

Appendix 2. The Proof of Proposition 2.1

Let $\{f_i\}_{i \geqq 1}$ be a sequence of holomorphic function in the unit ball $B_1(o) \subset C^2$ with $\lim_{i \to \infty} f_i = f \not\equiv 0$. We want to prove that for $\alpha < \inf_{x \in B_{\frac{1}{2}}(o)} \{\alpha_x(f)\}$,

$$\lim_{i \to \infty} \int_{B_{\frac{1}{2}}(o)} \frac{dV}{|f_i|^{2\alpha}} = \int_{B_{\frac{1}{2}}(o)} \frac{dV}{|f|^{2\alpha}} \tag{A.2.1}$$

where dV is the standard euclidean volume form.

As before, we denote by $Z(f)$ the zero locus of f and $Z(f)_{\text{red}}$ the sum of distinct irreducible components in $Z(f)$. Without losing the generality, we may assume that $Z(f)_{\text{red}}$ is smooth outside the origin o. Then $\alpha_x(f) = \dfrac{1}{m_x}$ for $x \neq 0$, where m_x is the multiplicity in $Z(f)$ of the irreducible component of $Z(f)_{\text{red}}$ containing x, $m_x = 0$ if $x \notin Z(f)_{\text{red}}$. On the other hand, m_x is the Lelong number of (1,1)-positive current $\partial\bar{\partial} \log|f|^2$ at x. By the estimate in [TY], for any small neighborhood U of o, and any $\beta < \min_{x \in B_{\frac{1}{2}}(o)} \{\alpha_x(f)\}$, there is a constant $C_{U,\beta}$ independent of i such that

$$\int_{B_{\frac{1}{2}}(o) \setminus U} \frac{dV}{|f_i|^{2\beta}} \leqq C_{U,\beta} \tag{A.2.2}$$

It implies that

$$\lim_{i \to \infty} \int_{B_{\frac{1}{2}}(o) \setminus U} \frac{dV}{|f_i|^{2\beta}} = \int_{B_{\frac{1}{2}}(o) \setminus U} \frac{dV}{|f|^{2\alpha}} \tag{A.2.3}$$

therefore, by the famous Fatou's lemma, in order to prove (A.2.1), it suffices to find a small neighborhood U of o such that

$$\lim_{i \to \infty} \int_U \frac{dV}{|f_i|^{2\alpha}} \leqq \int_U \frac{dV}{|f|^{2\alpha}} \tag{A.2.4}$$

Lemma A.2.1. *Suppose that for sequence $\{F_i\}_{i \geqq 1}$ of holomorphic functions in a neighborhood of o with $\lim_{i \to \infty} F_i = F \not\equiv 0$, there is a constant C_α independent of i such that for $\alpha < \alpha_o(F)$*

$$\int_U \frac{dV}{|F_i|^{2\alpha}} \leqq C_\alpha < \infty \tag{A.2.5}$$

where U is a fixed small neighborhood of o. Then (A.2.4) is valid.

Proof. Since $\alpha < \alpha_o(f)$, by taking U smaller if necessary and using Proposition A.1.1, we can choose local coordinates (z_1, z_2) at o such that we have the following expansion of f

$$\begin{aligned} f(z_1, z_2) &= z_1^{i_1} z_2^{j_2} \prod_{\nu=1}^{n(f)} (z_1^{j_1} + \lambda_\nu z_2^{i_2})^{p_\nu} + f_R(z_1, z_2) \\ &= f_\Delta(z_1, z_2) + f_R(z_1, z_2) \end{aligned} \tag{A.2.6}$$

where

(1) $i_1, j_2, p_\nu < \dfrac{1}{\alpha}$, $\nu = 1, 1, \ldots, n(f)$.

(2) Δ is a line segment in the first quadrant R_+^2 of R^2 and f_Δ contains exactly terms $z_1^k z_2^l$ in the expansion of f with $(k, l) \in \Delta$.

(3) For any term $z_1^k z_2^l$ in the expansion of f_R (k, l) lies on the right side of the line in R^2 containing Δ.

(4) $\alpha < r(N(f))^{-1}$, where $r(N(f))$ is the remotedness of the associated Newton polyhedron of f in local coordinates (z_1, z_2).

Define $\varepsilon_i = K \max_{\bar{U}} \{|f_i - f|\}$, where $K \geqq 2$ is a constant determined later. Then

$$\lim_{i\to\infty} \int_U \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.7}$$

$$\leqq \lim_{i\to\infty} \left\{ \frac{1}{(1-\varepsilon_i)^{2\alpha}} \int_U \frac{dV}{|f|^{2\alpha}} + \int_{\substack{|f|\leqq\varepsilon_i \\ U}} \frac{dV}{|f_i|^{2\alpha}} \right\}$$

$$= \int_U \frac{dV}{|f|^{2\alpha}} + \lim_{i\to\infty} \int_{\substack{|f|\leqq\varepsilon_i \\ U}} \frac{dV}{|f_i|^{2\alpha}}$$

Therefore, it suffices to prove that the last integral in (A.2.7) tends to zero as i goes to infinity.

We may take U to be $\{(z_1, z_2) \in C^2 \,||z_1| \leqq \delta^{i_2}, |z_2| \leqq \delta^{i_1}\}$. In the following, all integrals are taken on the subsets of U as specified.

We decompose the last integral in (A.2.7) into three parts $J_{i1}(\varepsilon_i)$, $J_{i2}(\varepsilon_i)$, $J_{i3}(\varepsilon_i)$ and estimate them individually. First we deal with $J_{i1}(\varepsilon_i)$. For that, we put $z_2 = w^{j_1}$, $z_2 = w^{j_2}\xi$, $m = i_1 i_2 + j_1 j_2 + i_2 j_1 \sum_{\nu=1}^{l} p_\nu$, $\delta_i = \varepsilon_i^{\frac{1}{m}}$, then

$$J_{i1}(\varepsilon_i) = \int_{\substack{|f|\leqq\varepsilon_i \\ |z_1|^{j_1}\leqq|z_2|^{j_2} \\ \sqrt[j_1]{|z_2|}\geqq\delta_i}} \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.8}$$

$$= j_1 \int_{\substack{|f|\leqq\varepsilon_i \\ \delta_i\leqq|w|,|\xi^l|\leqq 1}} \frac{|w|^{2j_1-2+2i_2} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{|f_i(w^{j_2}\xi, w^{j_1})|^{2\alpha}}$$

$$= j_1 \int_{\substack{|f|\leqq\varepsilon_i \\ \delta_i\leqq|w|,|\xi|\leqq 1}} \frac{|w|^{2(j_1+i_2-1-m\alpha)} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{|\xi|^{j_1} \prod_{\nu=1}^{l} (\xi + \lambda_\nu)^{p_\nu} + w^{-m} f_R(w^{j_2}\xi, w^{j_1}) + (f_i - f)|^{2\alpha}}$$

By the definitions of δ_i and f_R, for $\delta_i \leqq |w| \leqq \delta$

$$|w^{-m} f_R(w^{i_2}\xi, w_{j_1}) + w^{-m}(f_i - f)| \leqq C\delta + K^{-1} \tag{A.2.9}$$

where C is a constant depending only on f_R. Since $r(N(f)) = \dfrac{m}{j_1 + j_2}$, we have that $\alpha i_1 < 1$, $p_\nu \cdot \alpha < 1$ and $m\alpha - j_1 + 1 - i_2 < 1$. Thus if we choose K sufficiently large and δ sufficiently small, by Fubini theorem and (A.2.8), (A.2.9), one can easily see $\lim_{i\to\infty} J_{i1}(\varepsilon_i) = 0$. Similarly, one has $\lim_{i\to\infty} J_{i2}(\varepsilon_i) = 0$. Note that

$$J_{i2} = \int_{\substack{|f|\leqq\varepsilon_i \\ |z_1|^{j_1}\leqq|z_2|^{j_2} \\ \sqrt[j_2]{|z_1|}\geqq\delta_i}} \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.10}$$

Next, we estimate $J_{i3}(\varepsilon_i)$, which is dominated by the following integral

$$I_{\varepsilon_i}(f_i, \alpha) = \int_{\substack{|z_1|\leqq\delta_i^{j_2} \\ |z_2|\leqq\delta_i^{j_1}}} \frac{dV}{|f_i|^{2\alpha}} \left(\delta_i = \varepsilon_i^{\frac{1}{m}}\right) \tag{A.2.11}$$

Define holomorphic functions F_i, F by

$$F_i(z_1, z_2) = \delta_i^{-\frac{m}{2}} f_i\left(\delta_i^{\frac{i_2}{2}} z_1, \delta_i^{\frac{i_1}{2}} z_2\right)$$

$$F(z_1, z_2) = f_\Delta(z_1, z_2) \tag{A.2.12}$$

For i sufficiently large, those functions are well-defined on U with $\lim_{i\to\infty} F_i = F$ and $\alpha < \alpha_0(F) = r(N(f))^{-1}$. By the definition in (A.2.11) and (A.2.12), we have

$$I_{\varepsilon_i}(f_i, \alpha) = \delta_i^{\frac{i_2 + j_1 - m\alpha}{2}} I_{\sqrt{\varepsilon_i}}(F_i, \alpha) \tag{A.2.13}$$

It follows from our assumption of the lemma that $\lim_{i\to\infty} I_{\varepsilon_i}(f_i, \alpha) = 0$. Then, the lemma is proved.

It remains to verify the assumption in Lemma A.2.1. We will complete it by induction. For simplicity of notations, we assume that $F_i = f_i$, $F = f$. By the proof of Proposition A.1.1, there is a local coordinate system (z_1, z_2) such that either $\alpha_0(f) = r(N(f))^{-1}$, where $r(N(f))$ is the remotedness of the associated Newton polyhedron $N(f)$, or $f(z_1, z_2) = z_1^{i_1} z_2^{j_1} + f_R(z_1, z_2)$ with $\alpha_0(f) = i_1^{-1} \leqq j_1^{-1}$ and $pl + l > pi_1 + j_1$ for any term $z_1^k z_2^l$ in f_R, where p is a sufficiently large integer. First we assume that f is in the second case. Then we can write $f_i = f_{iL} + z_1^{i_1} z_2^{j_1} + f_{iR}$, where f_{iL} consists of all terms $z_1^k z_2^l$ in f_i with $pk + l < pi_1 + j_1$. By the holomorphic transformation of form $(z_1, z_2) \to (z_1 + \sum_{k \geq 1} z_2^k, z_2)$, we may further assume that f_{iL} does not have the term $z_1^{i_1 - 1} z_2^l$ with $l \geqq j_1 + 1$. Note that $\lim_{i\to\infty} f_{iL} = 0$, $\lim_{i\to\infty} f_{iR} = f_R$. Furthermore, by holomorphic transformation, we may assume that each f_{iL} does not contain any term $z_1^{i_1 - 1} z_2^l$ with $l > j_1$.

If $f_{iL} = \sum_{pk + l < pi_1 + j_1} a_{kl}(i) z_1^k z_2^l$, we define

$$\varepsilon_i = K \max\left\{|a_{kl}(i)|^{\frac{1}{pi_1 + j_1 - pk - l}}\right\} \tag{A.2.14}$$

where K is a large constant independent of i. We decompose

$$\begin{aligned}\int_U \frac{dV}{|f_i|^{2\alpha}} &= \left(\int_{\substack{|z_1| \leqq |z_2|^p \\ |z_2| \geqq \varepsilon_i}} + \int_{\substack{|z_1| \leqq |z_2|^q \\ |z_2| \geqq \varepsilon_i^p}} + \int_{\substack{|z_1| \leqq \varepsilon_i^q \\ |z_2| \geqq \varepsilon_i}} \right) \frac{dV}{|f_i|^{2\alpha}} \\ &= J_{1\alpha}(f_i, \varepsilon_i) + J_{2\alpha}(f_i, \varepsilon_i) + I_{\varepsilon_i}(f_i, \alpha)\end{aligned} \tag{A.2.15}$$

It is easy to show (cf. the proof of Lemma A.2.1) that both $J_{1\alpha}(f_i, \varepsilon_i)$ and $J_{2\alpha}(f_i, \varepsilon_i)$ are uniformly bounded if K is sufficiently large. Put $g_i = \varepsilon_i^{-(pi_1 + j_1)} f_i(\varepsilon_i^p z_1, \varepsilon_i z_2)$. Without losing generality, we may assume that $\lim_{i\to\infty} g_i = g$. Obviously, the holomorphic function g is the sum of $z_1^{i_1} z_2^{j_1}$ and some monomials $b_{kl} z_1^k z_2^l$ with $pk + l < pi_1 + j_1$.

Claim 1. For any point x in $B_1(o)$, we have $\alpha_x(g) \geqq \alpha_o(f)$.

We may take x to be o since the translations on C^2 preserve the property that g is a sum of $z_1^{i_1} z_2^{j_1}$ and some monomials $b_{kl} z_1^k z_2^l$ with $pk + l < pi_1 + j_1$ and $k < i_1 - 1$ or $k = i_1 - 1$, $l \leqq j_1$). Let $N_o(g)$ be the Newton polyhedron associated to g and coordinates (z_1, z_2), let Δ' be the line segment in $\partial N_o(g)$ intersecting with the diagonal line $\{x = y\}$ in R^2. Define

$$g_{\Delta'} = \sum_{(k,l) \in \Delta'} b_{kl} z_1^k z_2^l \text{ where } g = \sum_{pk + l \leqq pi_1 + j_1} b_{kl} z_1^k z_2^l \tag{A.2.16}$$

Then we can write

$$g_{\Delta'} = z_1^{i'_1} z_2^{j'_1} \prod_{v=1}^{n(g)} (z_1^{j'_1} + \lambda'_v z_2^{i'_2})^{p'_v} \tag{A.2.17}$$

By the proof of Proposition A.1.1, we have

$$\alpha_o(g) \geqq \left(\max_{1 \leqq v \leqq n(g)} \{r(N_o(g)), i'_1, j'_2, p'_v\} \right)^{-1} \tag{A.2.18}$$

On the other hand, we may assume p so large that there is no integer pair (k, l) in R^2 with $pk + l < pi_1 + j_1$, $k > i_1$ and $k \geqq l$. In particular, it implies that $r(N_o(g))$, $i'_1, j'_2, p'_v \leqq i_1$. It follows that $\alpha_o(g) \geqq \alpha_o(f)$. The claim is proved.

We also observe that $\mathrm{mult}_o(g_{\Delta'}) \leqq i_1 + j_1$ and $\alpha_o(g) > \alpha_o(f)$ whenever $\mathrm{mult}_o(g) = i_1 + j_1$. Thus by induction on $\alpha_o(f)$, i_1, $i_1 + j_1$, etc., we can verify the assumption of Lemma A.2.1 in this special case. Note that i_1 and $i_1 + j_1$ are determined by the lower endpoint of the line segment Δ (cf. (A.2.6)) of $\partial N(f)$ for a more coordinate system.

Next, we may assume that in the local coordinates (z_1, z_2), $\alpha_0(f) = r(N(f))^{-1}$ for the associated Newton polyhedron $N(f)$. Let Δ be line segment in $\partial N(f)$ intersecting with $\{x = y\}$ in R^2 with the properties stated as in (A.2.6) of the proof of Lemma A.2.1. Note that $n(f) \geqq 1$. Furthermore, if $j_1 \geqq 2$, then f_Δ does not have any monomials $z_1^k z_2^l$ with $k = i_1 + j_1 \sum_{v=1}^{n(f)} p_v - 1$; if $j_1 = 1$, by the transformation of form $(z_1, z_2) \to (z_1 + cz_2^{l_2}, z_2)$ and using the fact that i_1, j_2 and all p_v are less than $r(N(f))^{-1}$, we may also assume that f_Δ does not have any monomial $z_1^k z_2^l$ with $k = i_1 + j_1 \sum_{v=1}^{n(f)} p_v - 1$.

We decompose $f_i = f_{iL} + f_\Delta + f_{iR}$, where f_{iL} is the part of f_i consisting of all terms $z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and L_Δ denotes the defining equation of the line containing Δ. By scaling, we may take U to be $D_1 \times D_1$ in C^2, where $D_r (r > 0)$ denotes the disk in C of radius r, and f to be f_Δ. It follows that $\lim_{i \to \infty} f_{iR} = 0$.

Claim 2. There are local biholomorphisms $\phi_i = D_{\frac{1}{4}} \times D_{\frac{1}{4}} \to D_1 \times D_1$ such that (i) converge uniformly to the identity as i goes to infinity; (ii) the Taylor expansions of $f_i \circ \phi_i$ at o do not contain terms $z_1^k z_2^l$ with either $L_\Delta(k, l) < 0$ and $k > k_1 = i_1 + j_1 \sum_{v=1}^{l} p_v$ or $k = k_1 - 1$, $l > j_2$ and $L_\Delta(k, l) \neq 0$; (iii) $\lim_{i \to \infty} f_i \circ \phi_i = f$ on $D_{\frac{1}{4}} \times D_{\frac{1}{4}}$.

Note that (k_1, j_2) is the lower end point of Δ with $k_1 \geqq j_2$. We define ϕ_i by equations

$$\phi'_i(z_1, z_2) = \left(z_1, z_2 + \eta_i + \sum_{k=1}^{n_1} \sum_{l=1}^{j_2 - 1} b_{kl}(i) z_1^k z_2^l\right) \tag{A.2.19}$$

where n_1 is a large integer. Then one computes

$$\begin{aligned} &z_1^s \left(z_2 + \eta_i + \sum_{k=1}^{n_1} \sum_{l=0}^{j_2 - 1} b_{kl}(i) z_1^k z_2^l\right)^t \\ &= z_1^s (z_2 + \eta_i)^t + t z_1^s \sum_{k=1}^{n_1} z_1^k \sum_{l=0}^{j_2 - 1} b_{kl}(i) z_2^l (z_2 + \eta_i)^{t-1} + O(|b|^2) \end{aligned} \tag{A.2.20}$$

where $O(|b|^2)$ denotes a quantity bounded by $C \sum_{k,l \geqq 0} |b_{kl}|^2$. Let $f_i = \sum_{k,l \geqq 0} a_{kl}(i) z_1^k z_2^l$ and $f_i \circ \phi_i = \sum_{k,l \geqq 0} c_{kl}(i) z_1^k z_2^l$ be the Taylor expansions, then $\lim_{i \to \infty} a_{kl}(i) = 0$ whenever $L_\Delta(k, l) \neq 0$. Moreover, for any (k, l) with $l < j_2$ and $n_1 > k > k_1$, it follows from (A.2.20) that

$$c_{kl}(i) = \sum_{s,t \geqq 0} a_{st}(i) t \sum_{j=0}^{l} b_{k-sj}(i) \binom{t-1}{j} \eta_i^{t-j-1} + O(|b|^2) + O(|a|) \tag{A.2.21}$$

where $O(|a|)$ denotes a quantity bounded by $C \sum_{L_\Delta(k,l) \neq 0} |a_{kl}|$ and we let b_{k-sj} be zero if $k - s \leqq 0$. By Cauchy formula, one can show that those $(\frac{1}{2})^{s+t} a_{st}(i)$ with $L_\Delta(s, t) \neq 0$ converge to zero uniformly as i goes to infinity. We choose $\eta_i \leqq \frac{1}{4}$ satisfying

$$\lim_{i \to \infty} \eta_i = 0, \quad \sup_{L_\Delta(s,t) \neq 0} \left\{\left|\left(\frac{1}{2}\right)^{s+t} a_{st}(i)\right|\right\} \ll \eta_i \tag{A.2.22}$$

By either using Implicit Function Theorem or an iteration, it follows from (A.2.21) that there are $\{b_{kl}(i)\}$ with $\lim_{i \to \infty} b_{kl}(i) = 0$ such that $c_{kl}(i) = 0$ for those (k, l) with $k_1 < k < n_1$ and $l < j_2$. We use these $\{b_{kl}(i)\}$ in the definition (A.2.19) of ϕ'_i, then these ϕ'_i satisfy (i), (iii) in the statement of Claim 2 and the Taylor expansions of $f_i \circ \phi'_i$ do not contain terms $z_1^k z_2^l$ with $n > k > k_1$, $l \leqq j_2$. Next we construct a local biholomorphisms ϕ''_i of torm $(z_1, z_2) \to (z_1 + \sum_{k \geqq 1} d_k z_2^k, z_2)$ to eliminate terms $z_1^{k_1 - 1} z_2^l$ in $f_i \circ \phi'_i$ with $l > j_2$ and $L_\Delta(k_1 - 1, l) \neq 0$. Then our biholomorphisms ϕ_i are the compositions $\phi'_i \circ \phi''_i$. The claim is proved.

In particular, Claim 2 implies that the Jacobians $\mathrm{Jac}(\phi_i)$ of ϕ_i are uniformly bounded on $D_{\frac{1}{2}} \times D_{\frac{1}{2}}$. Therefore, it suffices to prove that $\int_{D_{\frac{1}{2}} \times D_{\frac{1}{2}}} |f_i \circ \phi_i|^{-2\alpha} dV$ are uniformly bounded. For simplicity, we still denote $f_i \circ \phi_i$ by f_i. Thus each f_{iL} does not contain any term $z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and either $k > k_1$ or $k = k_1 - 1$, $l > j_2$.

Since $\lim_{i \to \infty} f_{iR} = 0$, without losing generality, we may assume that

$$|f_{iR}|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \leqq \delta \quad \text{for all } i \tag{A.2.23}$$

where $\delta > 0$ is a sufficiently small number determined later.

Define $m = i_1 i_2 + j_1 j_2 + (\sum_{\nu=1}^{l} p_\nu) i_2 j_1$ and

$$\varepsilon_i = K \max\{|a_{kl}(i)|^{\frac{1}{m - i_2 k - j_1 l}} | L_\Delta(k, l) < 0\} \tag{A.2.24}$$

Lemma A.2.2. *Let $h = \sum_{k,l \geqq 0} b_{kl} z_1^k z_2^l$ be a holomorphic function on $D_1 \times D_1 \subset C^2$ such that $b_{kl} = 0$ if $L_\Delta(k, l) < 0$. Then we have*

(i) *for $|\xi| \leqq 1$, $|w|^{j_1} \leqq \frac{1}{2}$, $|w^{i_2}\xi| \leqq \frac{1}{2}$,*

$$|h(w_{i_2}\xi, w^{j_1})| \leqq (2|w|)^m |h|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \tag{A.2.25}$$

(ii) *for $|\xi| \leqq 1$, $|w|^{i_2} \leqq \frac{1}{2}$, $|w^{j_1}\xi| \leqq \frac{1}{2}$,*

$$|h(w^{i_2}\xi, w^{j_1}\xi)| \leqq (2|w|)^m |h|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \tag{A.2.26}$$

Proof. We just prove (i) here. The other case is analogous. For any fixed ξ with $|\xi| \leqq 1$, by our assumption on h_1, the function $w^{-m} h(w^{i_2}\xi, w^{j_1})$ is holomorphic in the domain $E_\xi = \{|w|^{j_1} \leqq \frac{1}{2}, |w^{i_2}\xi| \leqq \frac{1}{2}\}$. Thus maximum principle implies that for

$$\begin{aligned} \sup_{E_\xi} |w^{-m} h(w_{i_2}\xi, w^{j_1})| &\leqq \sup_{\partial E_\xi} |w^{-m} h(w^{i_2}\xi, w^{j_1})| \\ &\leqq 2^m |h|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \end{aligned} \tag{A.2.27}$$

The inequality (A.2.20) follows from it. Similarly, we can prove (A.2.21).

We then compute

$$\int_{D_{\frac{1}{2}} \times D_{\frac{1}{2}}} \frac{dV}{|f_i|^{2\alpha}} = \left(\int_{\substack{i_2\sqrt{|z_1|} \leqq \sqrt[j_1]{|z_2|} \\ \varepsilon_i \leqq \sqrt[j_1]{|z_2|}}} + \int_{\substack{i_2\sqrt{|z_1|} \geqq \sqrt[j_1]{|z_2|} \\ i_2\sqrt{|z_1|} \geqq \varepsilon_i}} + \int_{\substack{|z_1| \leqq \varepsilon_i^{i_2} \\ |z_2| \leqq \varepsilon_i^{j_1}}} \right) \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.28}$$

$$\begin{aligned} &\leqq j_1 \int_{\substack{\varepsilon_i \leqq |w| \leqq (\frac{1}{2})^{\frac{1}{j_1}} \\ |\xi| \leqq 1, |w^{i_2}\xi| \leqq \frac{1}{2}}} \frac{|w|^{2i_2 + 2j_1 - 2 - 2m\alpha} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{\left| \xi^{i_1} \prod_{\nu=1}^{l} (\xi + \lambda_\nu)^{p_\nu} + w^{-m}(f_{iL}(w^{i_2}\xi, w^{j_1}) + f_{iR}(w^{i_2}\xi, w^{j_1})) \right|^{2\alpha}} \\ &+ i_2 \int_{\substack{\varepsilon_i \leqq |w| \leqq (\frac{1}{2})^{\frac{1}{j_1}} \\ |\xi| \leqq 1, |w^{i_2}\xi| \leqq \frac{1}{2}}} \frac{|w|^{2i_2 + 2j_1 - 2 - 2m\alpha} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{\left| \xi^{i_2} \prod_{\nu=1}^{l} (1 + \lambda_\nu \xi)^{p_\nu} + w^{-m}(f_{iL}(w^{i_2}, w^{j_1}\xi) + f_{iR}(w^{i_2}, w^{j_1}\xi)) \right|^{2\alpha}} \\ &+ I_{\varepsilon_i}(f_i, \alpha) \\ &= J_{1\varepsilon_i}(f_i, \alpha) + J_{2\varepsilon_i}(f_i, \alpha) + I_{\varepsilon_i}(f_i, \alpha) \end{aligned}$$

Since the nonzero roots $\lambda_1, \lambda_2, \ldots, \lambda_l$ are distinct, we have

$$\lambda = \min\{|\lambda_\mu - \lambda_\nu|, |\lambda_\mu - \lambda_\nu|^{-1}, |\lambda_\mu|, |\lambda_\nu|, |\lambda_\mu|^{-1}, |\lambda_\nu|^{-1} | 1 \leqq \mu < \nu \leqq l\} > 0 \tag{A.2.29}$$

By (A.2.19) and Lemma A.2.2, for $(\frac{1}{2})^{\frac{1}{j_1}} \geqq |w| \geqq \varepsilon_i$, $|\xi| \leqq 1$ and $|w^{i_2}\xi| \leqq \frac{1}{2}$,

$$|w^{-m}(f_{iL}(w^{i_2}\xi, w^{j_1}) + f_{iR}(w^{i_2}\xi, w^{j_1}))| \leqq n_\Delta \cdot K^{-1} + 2^m \delta \tag{A.2.30}$$

where n_Δ is the number of the integer pairs (k, l) with $L_\Delta(k, l) < 0$. Choose K, δ such that $n_\Delta K^{-1} < \frac{1}{10}\lambda^m$ and $2^{m+1}\delta < \frac{1}{10}\lambda^m$. Then there is a constant $C_{\lambda,\alpha}$, depending only on λ, α, such that $J_{1\varepsilon_i}(f_i, \alpha) \leqq \frac{1}{2}C_{\lambda,\alpha}$. Similarly, $J_{2\varepsilon_i}(f_i, \alpha) \leqq \frac{1}{2}C_{\lambda,\alpha}$. It follows that

$$\int_{D_{\frac{1}{2}} \times D_{\frac{1}{2}}} \frac{dV}{|f_i|^{2\alpha}} \leqq (i_2 + j_1)C_{\lambda,\alpha} + I_{\varepsilon_i}(f_i, \alpha) \tag{A.2.31}$$

Put $g_i = \varepsilon_i^{-m} g_i(\varepsilon_i^{i_2} z_1, \varepsilon_i^{j_1} z_2)$, then by taking a subsequence, we may assume that $\lim_{i\to\infty} g = g$. Note that g is a sum of $f = f_\Delta$ and some monomials $b_{kl}z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and $k \leqq k_1 = i_1 + j_1 \sum_{\nu=1}^{l} p_\nu$, $l \leqq j_2$ if $k = k_1 - 1$.

Lemma A.2.3. *Let f, g be given as above. Then*

$$\inf_{x \in D_1 \times D_1} \{\alpha_x(g)\} \geqq \alpha_o(f) \tag{A.2.32}$$

Proof. Since f does not contain any monomial $z_1^{k_1 - 1}z_2^l$, any composition of g with a translation in C^2 is still a sum of f_Δ and some monomials $b_{kl}z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and $k \leqq k_1$, $l \leqq j_2$ if $k = k_1 - 1$. Hence, we may take x to be the origin in (A.2.32). Recall

$$f_\Delta = z_1^{i_1} z_2^{j_2} \prod_{\nu=1}^{n(f)} (z_1^{i_1} + \lambda_\nu z_2^{i_2})^{p_\nu},\ i_2 \geqq j_1$$

Let Δ' be the line segment in $\partial N(g)$ intersecting with the line $\{x = y\}$ in R^2 and $g_{\Delta'}$ be the polynomial consisting of all monomials $b_{kl}z_1^k z_2^l$ of g with $(k, l) \in \Delta'$. As above, we can write

$$g_{\Delta'} = z_1^{i'_1} z_2^{j'_1} \prod_{\nu=1}^{n(g)} (z_1^{j'_1} + \lambda'_\nu z_2^{i'_2})^{p'_\nu} \tag{A.2.33}$$

Then $i'_1, j'_2 \leqq r(N(f)) = \alpha_o(f)^{-1}$ and $r(N(g)) \leqq r(N(f))$.

By the proof of Proposition A.1.1, we have the estimate

$$\alpha_o(g) \geqq \min\{(i'_1)^{-1}, (j'_2)^{-1}, r(N(g))^{-1}, (p'_\nu)^{-1}\} \tag{A.2.34}$$

Suppose that one of p'_ν, say p'_1 for simplicity, is greater than $\alpha_0(f)^{-1}$. By (A.2.32) and the definition of g, we have

$$i'_1 + j'_1 \sum_{\nu=1}^{n(g)} p'_\nu \leqq i_1 + j_1 \sum_{\nu=1}^{n(f)} p_\nu \leqq 2r(N(f)) \tag{A.2.35}$$

where the equality holds iff $j_2 = i_1 + j_1 \sum_{\nu=1}^{n(f)} p_\nu = r(N(f))$. It follows that $j'_1 = 1$ and $i'_2 \geqq j'_1 = 1$.

By local holomorphic transformations at o, we may further assume that $i'_2 \geqq \dfrac{i_2}{j_1}$.

In case $j_1 = 1$, we claim that $i'_2 = i_2$. Suppose that $i'_2 \geqq i_2 + 1$. Note that g contains the monomial $z_1^{i'_1} z_2^{i'_2}$ with $i'_2 = j'_2 + i'_2 \sum_{\nu=1}^{n(f)} p'_\nu$. Then by $L_\Delta(i'_1, j'_2 + i'_2 \sum_{\nu=1}^{n(g)} p'_\nu) \leqq 0$, we have

$$\begin{aligned} i_1 i_2 + j_2 + i_2 \sum_{\nu=1}^{n(f)} p_\nu &\geqq i_2 i'_1 + \left(j'_2 + i'_2 \sum_{\nu=1}^{n(g)} p'_\nu\right) \\ &\geqq i'_2 p'_1 > i'_2 \alpha_o(f) \\ &= \frac{i'_2}{i_2 + 1}\left(i_1 i_2 + j_2 + i_2 \sum_{\nu=1}^{n(f)} p_\nu\right) \end{aligned}$$

A contradiction! Thus $i'_2 = i_2$. We define a new local coordinate system $(\tilde{z}_1, \tilde{z}_2)$, by setting $\tilde{z}_1 = z_1 + \lambda'_1 z_2^{i_2}$, $\tilde{z}_2 = z$, then

$$f_\Delta(\tilde{z}_1, \tilde{z}_2) = (\tilde{z}_1 - \lambda'_1 \tilde{z}_2^{i_2})^{i'_1} \tilde{z}_2^{j_2} \prod_{\nu=1}^{n(f)} (\tilde{z}_1 + (\lambda_\nu - \lambda'_1)\tilde{z}_2^{i_2})^{p_\nu}$$

If either none of λ_ν is equal to λ_1' or some λ_{ν_0} is equal to λ_1' and $i_1 + j_2 + \sum_{\nu \neq \nu_0} p_\nu > p_{\nu_0}$, the function $f_\Delta(\tilde{z}_1, \tilde{z}_2)$ has the same properties of f_Δ in (z_1, z_2). The above arguments shows that $\alpha_o(g) \geqq \alpha_o(f)$. If some λ_{ν_0} is equal to λ_1', $i_1 + j_2 + \sum_{\nu \neq \nu_0} p_\nu \leqq p_{\nu_0}$, then by the proof of the Proposition A.1.1, we have

$$\alpha_o(g) \geqq p_{\nu_0}^{-1} \geqq \alpha_0(f)$$

The lemma is proved in case $j_1 = 1$. The proof of the case $j_2 \geqq 2$ is analogous and a little bit complicated. We omit the details and make the following remarks. If $\alpha_o(g) < \alpha_o(f)$, then one can choose $\lambda_{k_1}, \ldots, \lambda_{k_n} \neq 0$ with $k_n > k_{n-1} > \ldots > k_1 \geqq 2$ such that in local coordinates $(\tilde{z}_1, \tilde{z}_2)$ with $\tilde{z}_1 = z_1 + \sum_{s=1}^n \lambda_k z_2^{k_s}$ and $\tilde{z}_2 = z_2$, the associated Newton polyhedron $N_1(g)$ lies entirely outside the triangle $D_{L'}$ defined by $\tilde{x}_1$-axis, $\tilde{x}_2$-axis and the line L' through $(\alpha_o(f)^{-1}, \alpha_o(f)^{-1})$ and an integer point (k', l') in $R_+ \times R_+$ with $k' > \alpha_0(f)^{-1}$ and $L_\Delta(k', l') \leqq 0$. That is, the polynomial $g(\tilde{z}_1, \tilde{z}_2)$ does not contain the monomials $z_1^k z_2^l$ with $(k, l) \in D_{L'}$. On the other hand, one can easily show that the triangle $D_{L'}$ contains strictly more integer points than the triangle D_Δ, defined by z_1-axis, z_2-axis and the line containing Δ, does. Moreover, the fact that $g(\tilde{z}_1, \tilde{z}_2)$ does not contain $\tilde{z}_1^k \tilde{z}_2^l$ with $(k, l) \in D_{L'}$ imposes sufficiently many independent equations on the coefficients in $g(z_1, z_2)$. In particular, it implies that $g(z_1, z_2) = f_\Delta(z_1, z_2)$. A contradiction! Therefore, we always have $\alpha_o(g) \geqq \alpha_o(f)$.

Let h be a holomorphic function in U, $x \in U$, we define a quantity $\mu_x(h)$ as follows. If there is a local coordinate system such that $\alpha_x(h) = r(N_x(f))^{-1}$ for the associated Newton polyhedron $N_x(f)$, we define $i_1(h_{\tilde{\Delta}})$ to be the smaller one of the x-component of the lower endpoint of $\tilde{\Delta}$ and the y-component of the upper endpoint of $\tilde{\Delta}$, where $\tilde{\Delta}$ is the line segment in $N_x(f)$ intersecting with $\{x = y\}$ in R^2. Then $\mu_x(h)$ is the infimum of such $i_1(h_{\tilde{\Delta}})$ among all positive local coordinate systems such that $\alpha_x(h)$ is the remotedness of the associated Newton polyhedron. Otherwise, there is a local coordinate system (z_1, z_2) such that $h = z_1^i h_1(z_1, z_2) + O((|z_1|^2 + |z_2|^2)^p)$, where $h_1(0, z_2) \neq 0$ and p is sufficiently large. Then we define $\mu_x(h) = i$.

Now we complete our verification of the assumption in Lemma A.2.1. Obviously, we have $\mu_x(g) \leqq \mu_o(f)$ for any $x \in D_1 \times D_1$ and in case $\mu_x(g) = \mu_o(f)$, for simplicity, say $x = o$, then by the assumption that $g_{\Delta'}$ does not have term $z_1^{k_1 - 1} z_2^l$ with $l > j_2$, we have $\alpha_0(g) \geqq \alpha_0(f) + \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a positive number depending only on the upper bound of $\alpha_o(f)^{-1}$ and $\text{mult}_o(f)$. Therefore by induction, one can easily see that the assumption in Lemma A.2.1 holds.

The proof of Proposition 2.1 is completed.

References

[AGV] Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differentiable maps, vol. II. Monogr. Math., **83**, Boston Basel Stuttgart, Birkhaüser 1988

[An] Anderson, M.: Ricci curvature bounds and Einstein metrics on compact manifolds. Preprint

[Au] Aubin, T.: Réduction du cas positif de l'équation de Monge-Ampère sur les variétés Kählerinnes compactes à la démonstration d'un intégalité. J. Funct. Anal. **57**, 143–153 (1984)

[Bai] Baily, W.: On the imbedding of V-manifolds in projective space. Am. J. Math. **79**, 403–430 (1957)

[Ban] Bando, S.: The K-energy map, almost Einstein-Kähler metrics and an inequality of the Miyaoka-Yau type. Tohoku Math. J. **39**, 231–235 (1987)

[Bi] Bishop, R.L., Crittenden, R.J.: Geometry of Manifolds, Pure and Applied Math., Vol. XV. New York-London: Academic Press 1964

[BM] Bando, S., Mabuchi, T.: Uniqueness of Einstein Kähler metrics modulo connected group actions. In: Oda T. (ed.) Alg. Geom., Sendai, Adv. Stud. Pure Math. **10**, Kinokunia, Tokyo and North Holland, Amsterdam, New York, Oxford, 1985

[BPV] Barth, W., Peters, C., Van de Ven, A.: Compact complex surfaces. Berlin-Heidelberg-New York: Springer 1984

[CE] Cheeger, J., Ebin, D.G.: Comparison theorems in Riemannian geometry. North-Holland Publishing Company, Amsterdam, 1975

[CGT] Cheeger, J., Gromov, M., Taylor, M.: Finite propagation speed, kernel estimates for

functions of the Laplace operator, and the geometry of complete Riemannian manifolds. J. Differ. Geom. **17**, 15–54 (1982)

[Cr] Croke, C.: Some isoperimetric inequalities and eigenvalue estimates. Ann. Sci. Ec. Norm. Super, Ser. **13**, 419–435 (1980)

[De] Demazure, M.: Surfaces de Del Pezzo. (Lecture Notes in Math., vol. 777, pp. 21–69). Berlin-Heidelberg-New York: Springer 1980

[Fu] Futaki, S.: An obstruction to the existence of Einstein-Kähler metrics. Invent. Math. **73**, 437–443 (1983)

[GH] Griffith, P., Harris, J.: Principles of Algebraic Geometry. New York: Wiley, 1978

[GLP] Gromov, M., Lafontaine, J., Pansu, P.: Structure metrique pour les varietes Riamanniennes. Nathen: Cedic/Fernand 1981

[GT] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Berlin-Heidelberg-New York: Springer 1977

[GW] Greene, R., Wu, H.: Lipschitz convergence of Riemannian manifolds. Pac. J. Math. **131**, 119–141 (1988)

[Ho] Hömander, L.: An introduction to complex analysis in several variables. Princeton: Van Nostrand 1973

[Jo] Jost, J.: Harmonic mappings between Riemannian manifolds. Proceedings of the center for mathematical analysis. Australian National Univ., Vol. 4, 1983

[KM] Kodaira, K., Morrow, J.: Complex manifolds. Holt-Rinehart and Winston, Inc. New York 1971

[KT] Kazdan, J., DeTurke, D.: Some regularity theorems in Riemannian geometry. Ann. Sci. Ec. Norm. Super, IV. Ser. **14**, 249–260 (1981)

[La] Laufer, H.: Normal two-dimensional singularities. Ann. Math. Stud. **71**, Princeton University Press 1971

[Le] Lelong, P.: Fonctions plurisousharmoniques et forms différentielles positives. New York: Gondon and Breach et Paris, Dunod 1969

[Li] Li, P.: On the Sobolev constant and the p-spectrum of a compact Riemannian manifols. Ann. Sci. Ec. Norm. Super, IV. Ser. **13**, 451–469 (1980)

[Ma] Matsushima, Y.: Sur la structure du group d'homeomorphismes analytiques d'une certaine varietie Kaehlerinne. Nagoya Math. J. **11**, 145–150 (1957)

[Na] Nakajima, H.: Hausdorff convergence of Einstein 4-manifolds. J. Fac. Sci. Univ. Tokyo **35**, 411–424 (1988)

[Si] Siu, Y.T.: The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. Ann. Math. **127**, 585–627 (1988)

[T1] Tian, G.: On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. Invent. Math. **89**, 225–246 (1987)

[T2] Tian, G.: A Harnack inequality for some complex Monge-Ampére equations. (To appear) J. Differ. Geom.

[TY] Tian, G., Yau, S.T.: Kähler-Einstein metrics on complex surfaces with $C_1(M)$ positive. Commun. Math. Phys. **112**, 175–203 (1987)

[Uh1] Uhlenbeck, K.: Removable singularities in Yang-Mills field. Commun. Math. Phys. **83**, 11–29 (1982)

[Uh2] Uhlenbeck, K.: Connections with L^p bounds on Curvature. Commun. Math. Phys. **83**, 31–42 (1982)

[Y1] Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation, *I**. Commun. Pure Appl. Math. **31**, 339–411 (1978)

[Y2] Yau, S.T.: On Calabi's conjecture and some new results in algebraic geometry. Proc. Natl. Acad. Sci. USA **74**, 1798–1799 (1977)

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Note added in proof

After submitting this paper, the author found V.N. Karpushkin's work on uniform estimates of oscillatory integrals in R^2 (J. of Soviet Math., **35**, 2809–2826 (1986)). A much simpler proof can be given for the main result in Appendix 2 by using this work, in particular, Theorem 3.1 in the above reference.

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On complex manifolds which can be compactified by adding finitely many points

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0. Introduction and statement of results

The following *compactification problem* arises naturally in attempts to classify complex manifolds: *Find sufficient conditions under which a given noncompact complex manifold can be compactified – i.e. exhibited as a Zariski-open subset of a compact complex space or, better yet, a complex projective variety.* By considering such a problem, one hopes to reduce the study of certain noncompact complex manifolds to that of compact ones, which are often easier to handle from the perspectives of intersection theory, analysis, and classification.

The compactification of arithmetic quotients of bounded symmetric domains was obtained by Andreotti-Grauert [AG], Satake [Sat], Baily-Borel [BB], and Ash-Mumford-Rapoport-Tai [AMRT]. More recently, Siu-Yau [SY], Nadel-Tsuji [NT], Mok [M1, M2], Mok-Zhong [MZ], and Yeung [Ye] compactified complete Kähler manifolds satisfying various curvature (as well as other) conditions. In this paper we shall consider the problem of compactifying strictly pseudoconcave complex manifolds. The open embedding of strictly pseudoconcave complex manifolds into compact complex manifolds was obtained previously in [Ros, AT, AS], and the goal of the present paper is to determine when the open embedding is actually Zariski-open. Our first result is the following.

Theorem 0.1 (Main Theorem). *Let M be a connected complex manifold of dimension $m \geqq 3$. Assume that*

1. *(Strict pseudoconcavity.) There exists a continuous proper map $\varphi: M \to [0, \infty)$ such that $-\varphi$ is strictly plurisubharmonic outside of a compact subset of M.*
2. *(Moishezon property.) There exist m meromorphic functions $f_1, \ldots, f_m$ on M which are algebraically independent over $\mathbf{C}$.*
3. *(Generalized affine-open cover.) M can be covered by Zariski-open sets U which are uniformized by Stein manifolds.*

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Then M is biholomorphic to $M^* - S$ where M^* is a compact complex Moishezon space (or equivalently an algebraic space, proper over $\mathbf{C}$ [Art]) and $S \subset M^*$ is a finite subset. In particular, M admits the structure of an algebraic space and has finite topological type.

It is worth noting that the hypotheses of Theorem 0.1 are purely complex analytic in nature (with the possible exception of condition (1), depending on one's terminology), and in particular make no reference to curvature, symmetry, or topological type. In this respect, Theorem 0.1 differs from previous compactification results.

Examples. Complex manifolds satisfying the hypotheses of Theorem 0.1 can be constructed as follows. Start off with any complete (i.e. proper over $\mathbf{C}$) abstract algebraic variety M^* possessing at worst isolated singularities and set $M = M^* - S$ where $S \subset M^*$ is any finite subset containing all singularities. Condition (2) obviously holds, as does condition (1) since φ can be constructed locally near each puncture point $s \in S$. Finally, condition (3) holds because every abstract algebraic variety admits by definition an affine (hence Stein) Zariski-open cover.

Another type of example is afforded by a complete Kähler manifold of finite volume and bounded negative sectional curvature. However, we appropriately postpone further discussion of this example until after the statement of Theorem 0.2 below, where we will be able to deduce quasiprojectivity.

In Section 5 we shall give a simple construction of a complex manifold M which satisfies conditions (1) and (2) of Theorem 0.1 and yet which is not of finite topological type; this implies that Theorem 0.1 is sharp in a certain sense.

Discussion of the Proof. The proof of Theorem 0.1 is rather technical, but follows the general line of reasoning introduced in [NT]: Let M be as in the statement Theorem 0.1. Because M is strictly pseudoconcave of dimension at least three, it embeds as an open subset of a compact complex manifold $\bar{M}$ [Ros]. As $\bar{M} - M$ is the pluripolar set of $-\varphi$, it has measure zero in $\bar{M}$. However, it is much more difficult to show that M is Zariski-open in $\bar{M}$, and for this we extend the methods of [NT] involving the L^2-Riemann-Roch Theorem and Kähler-Einstein geometry. The approach may be described very roughly as follows. Because the algebraic dimension of M equals m by assumption, $\bar{M}$ is Moishezon and is therefore bimeromorphically dominated by a projective algebraic manifold. One then finds a suitable analytic subvariety V of $\bar{M}$ such that both $M - V$ and $\bar{M} - V$ admit complete Kähler-Einstein metrics g and g' respectively. By the Schwarz lemma for volume forms it then follows that $(g')^m \leqq g^m$ on $M - V$. On the other hand, by the L^2-Riemann-Roch Theorem on $M - V$ together with the usual Riemann-Roch Theorem on $\bar{M}$, $\int_M g^m \leqq \int_M (g')^m$. Together these two inequalities imply that $(g')^m = g^m$ on $M - V$, and by the Kähler-Einstein property of the metrics it follows that $g' = g$ on $M - V$ and hence that $M - V = \bar{M} - V$. Finally, by "moving V around" one concludes that M is Zariski-open in $\bar{M}$. Because of the strict pseudoconcavity of M, one can blow $\bar{M} - M$ down to a finite set, and Theorem 0.1 follows.

Variants of the Main Theorem. We present below two alternative formulations of Theorem 0.1 which have the advantage over Theorem 0.1 of being sharp: Theorem 0.2 (resp. Theorem 0.3) gives a *precise characterization* of those noncompact complex manifolds that can be compactified by adding finitely many points and that admit quasiprojective (resp. abstract) algebraic structure (see Proposition 0.1 below).

Theorem 0.2 (First variant of the Main Theorem). *Let M be a connected complex manifold of dimension $m \geqq 3$. Assume that*

1. *There exists a continuous proper map $\varphi\colon M \to [0, \infty)$ such that $-\varphi$ is strictly plurisubharmonic outside of a compact subset.*
2. *There exists a holomorphic line bundle L over M such that $\bigcup_{\nu \geqq 1} H^0(M, L^\nu)$ separates points and forms local coordinates on M.*
3. *M can be covered by Zariski-open sets U which are uniformized by Stein manifolds.*

Then M is biholomorphic to a quasiprojective variety which can be compactified to an algebraic space by adding finitely many points.

Example. Let M be a complete Kähler manifold of finite volume and bounded negative sectional curvature. Assume that $\dim_{\mathbf{C}} M \geqq 3$. We will show that M satisfies the three hypotheses of Theorem 0.2. As in [SY] one can use Buseman functions to construct the required strictly plurisuperharmonic exhaustion (see also the final section of [NT]). Thus M satisfies (1). Since M has negative Ricci curvature one can use L^2 estimates of $\bar{\partial}$ to show that M satisfies condition (2) with $L = K_M$. Finally, since any complete simply connected Kähler manifold of seminegative sectional curvature is Stein, it follows that the universal cover of M is Stein and in particular that (3) holds. Theorem 0.2 now implies that M is biholomorphic to a quasiprojective variety which can be compactified to a complex space by adding finitely many points. We thus recover a strong form of the main result of [SY] in dimension greater than two.

Theorem 0.3 (Second variant of the Main Theorem). *Let M be a connected complex manifold of dimension $m \geqq 3$. Assume that*

1. *There exists a continuous proper map $\varphi\colon M \to [0, \infty)$ such that $-\varphi$ is strictly plurisubharmonic outside of a compact subset.*
2. *M can be covered by Zariski-open sets U which are uniformized by bounded domains of holomorphy in $\mathbf{C}^m$.*

Then M is biholomorphic to an abstract algebraic variety which can be compactified to an algebraic space by adding finitely many points.

Proposition 0.1 (The converse). *Let M be a smooth quasiprojective (resp. abstract) algebraic variety which can be compactified to a compact complex space by adding finitely many points. Then M satisfies conditions* (1)–(3) *of Theorem* 0.2 (*resp. conditions* (1) *and* (2) *of Theorem* 0.3).

Proof. Griffiths [Gri] proved that every (abstract) complex algebraic variety can be covered by Zariski-open sets which are uniformized by bounded contractible domains of holomorphy. This takes care of condition (2) of Theorem 0.3. Everything else is clear. QED

Thus Theorems 0.2 and 0.3 are sharp.

Acknowledgements. The results in this paper are based on techniques first introduced in [NT], and it is a pleasure to acknowledge here my intellectual debt to H. Tsuji, my co-worker in [NT]. I would also like to thank Y.-T. Siu for many valuable discussions over the past several years. Finally, I would like to thank the Institute for Advanced Study for its hospitality and financial support.

Conventions. A *subvariety* of a complex manifold is defined to be a closed analytic subset, not necessarily irreducible. An irreducible (possible noncompact) complex space is said to be *Moishezon* iff its algebraic dimension equals its dimension. An *abstract algebraic variety* is defined to be a reduced scheme of finite type over **C**. Unless we specify the contrary, complex manifolds are not assumed connected and algebraic varieties are not assumed irreducible.

This paper is organized as follows:

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1. Preliminary material

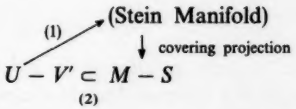
In this section we collect various results which will be needed in the proofs of Theorems 0.1–0.3.

1.1. Stein manifolds

Proposition 1.1. *Let M be a connected complex manifold and $S \subset M$ a nonempty closed subset such that $M - S$ is uniformized by a Stein manifold. Suppose that*

$S \subset V \cup V'$ where V and V' are analytic subvarieties of M such that V' has codimension $\geqq 2$ in M. Then in fact $S \subset V$.

Proof. Suppose, on the contrary, that $S \not\subset V$. Then there exists an open coordinate ball $U \subset M - V$ such that $U \cap S \neq \varnothing$. We note that $U - V'$ is simply connected since V' has codimension $\geqq 2$. Therefore the inclusion $U - V' \subset M - S$ lifts to a map into the universal cover of $M - S$ to give a commutative triangle



According to Lemma 1.1 below, the map (1) extends holomorphically to all of U. Therefore the inclusion (2) extends to all of U, which is absurd since $S \cap U \neq \varnothing$. This contradiction proves the proposition. QED

Lemma 1.1. *Let M be a complex manifold, $V \subset M$ an analytic subvariety of codimension $\geqq 2$, and N a Stein manifold. Then every holomorphic map*

$$M - V \to N$$

extends holomorphically to all of M.

Proof. In the special case $N = \mathbf{C}^n$ this lemma reduces to the Levi Extension Theorem. The general case follows from this special case since every Stein manifold can be embedded as a closed submanifold of some $\mathbf{C}^n$. QED

Proposition 1.2. *Let M be a Stein manifold and $\tilde{M} \to M$ a covering. Then $\tilde{M}$ is Stein*

This result is now classical and is due to K. Stein himself.

Proposition 1.3. *Let M be a Stein manifold and D a reduced effective divisor on M. Then $M - D$ is Stein.*

Proof. Let $L = [D]$ be the holomorphic line bundle over M associated to the divisor D, and let $s \in H^0(M, L) - \{0\}$ be a global holomorphic section whose zero locus is precisely D. Endow L with a smooth Hermitian metric (along the fibers) such that $|s| \leqq 1$ (pointwise on M). Denote by $c_1(L)$ the associated curvature form. Since M is Stein it admits a smooth strictly plurisubharmonic exhaustion φ. After replacing φ by $\chi(\varphi)$ where $\chi: \mathbf{R} \to \mathbf{R}$ is sufficiently convex and increasing, we can assume that $\partial\bar{\partial}\varphi + c_1(L) > 0$ (pointwise on M). Then $\varphi + \log \frac{1}{|s|^2}$ is a strictly plurisubharmonic exhaustion of $M - D$, as desired. As an aside, we remark that Docquier and Grauert have proved the more general result that every locally Stein open subset of a Stein manifold is Stein. QED

Proposition 1.4. *A given domain $\Omega \subset \mathbf{C}^m$ is a domain of holomorphy iff its universal cover $\tilde{\Omega}$ is Stein.*

Proof. On direction follows from Proposition 1.2. For the other direction we assume that $\tilde{\Omega}$ is Stein. A classical result of Oka characterizes domains of holomorphy as those domains which satisfy the Kontinuitätssatz (see [Siu]). But it is immediate that Ω satisfies the Kontinuitätssatz since its universal cover is Stein. QED

1.2. Moishezon manifolds

In this subsection we shall present an elementary result concerning divisors on Moishezon manifolds. Let M be a compact Moishezon manifold. As is well-known, M can be "blown-up" to a smooth complex projective variety. More precisely, there exists a holomorphic map $f: M' \to M$ from a smooth projective variety M' and a subvariety $C \subset M$ of codimension $\geqq 2$ such that

1. The map f induces a biholomorphism $M' - f^{-1}(C) \to M - C$.
2. The fibers $f^{-1}(c)$ have positive dimension for all $c \in C$.

Such a subvariety $C \subset M$ will be called a *center* for f.

Proposition 1.5. *Let M, M', C, f be as above, and let $p \in M - C$ be a given point. Then there exists a reduced effective divisor D on M such that*

1. *$C \subset D$, and*
2. *$p \notin D$.*

Proof. Let D' be an ample divisor in general position on M' whose support does not contain $f^{-1}(p)$, and set $D = f(\text{support of } D')$. Since D' is in general position, D has codimension 1 in M. Clearly (2) holds. As for (1), we note that every fiber $f^{-1}(c)$ $(c \in C)$ has positive dimension and therefore must intersect every ample divisor on M'. QED

1.3. Cusp forms and the uniqueness of algebraic structure on varieties of log-general type

Let M be a complex manifold and let $\psi \in H^0(M, K_M^{\otimes \nu})$ be a ν-canonical form on M for some integer $\nu > 0$.

Definition 1.1. We say that ψ is a cusp form on M provided that

$$\int_M (\psi \wedge \bar{\psi})^{\frac{1}{\nu}} < \infty \ .$$

Proposition 1.6. *Let M be a complex manifold, D a reduced effective divisor on M, and ψ a cusp form on $M - D$. Then ψ extends meromorphically to a pluricanonical form on M with strictly less than logarithmic poles along D.*

A proof can be found in [Sak]. Conversely, we have the following proposition.

Proposition 1.7. *Let M be a complex manifold and let ψ be a meromorphic section of $K_M^{\otimes v}$ over M. Assume that ψ has strictly less than logarithmic poles. Then*

$$\int_S (\psi \wedge \bar{\psi})^{\frac{1}{v}} < \infty \ .$$

for any compact subset S of M.

The proof of this proposition is trivial when the pole set of ψ has simple normal crossings, and one can always reduce to this case by employing Hironaka's resolution of singularities.

From the above considerations (and GAGA) we deduce the following. On a smooth algebraic variety all cusp forms are algebraic. A smooth connected algebraic variety of dimension m is of log-general type if and only if its ring of cusp forms has transcendence degree $m+1$ over $\mathbf{C}$.

Proposition 1.8. *Let U, U' be smooth quasiprojective varieties of the same dimension and let $f: U \to U'$ be a biholomorphism of U onto an open subset (in the classical topology) of U'. Assume that U' is of log-general type. Then f is a morphism of algebraic varieties.*

Proof. Thus we consider the case of an open inclusion $U \subset U'$. To show that this inclusion is a morphism of algebraic varieties it is sufficient to show that $V' \cap U$ is an algebraic subvariety of U for every algebraic subvariety V' of U'. Clearly we may assume further that V' has pure codimension 1. Let $\psi \not\equiv 0$ be a cusp form on U' whose zero set contains V'. Such ψ exists because U' is of log-general type. Then

$$\int_U (\psi \wedge \bar{\psi})^{\frac{1}{v}} \leqq \int_{U'} (\psi \wedge \bar{\psi})^{\frac{1}{v}} < \infty$$

which implies that the restriction of ψ to U is a cusp form on U and hence is algebraic on U. Therefore the zero set of ψ restricts to an algebraic subvariety of U. We conclude that $V' \cap U$ is an algebraic subvariety of U, as desired. QED

Similarly, we have

Proposition 1.9. *Let U, U' be smooth algebraic varieties of the same dimension and let $f: U \to U'$ be a biholomorphism of U onto U'. If either one of U, U' is of log-general type then so is the other, and f is an isomorphism of algebraic varieties.*

1.4. The Riemann-Roch inequality for the L^2-plurigenera of a complete Kähler manifold

The following Riemann-Roch inequality for the L^2-plurigenera of a complete Kähler manifold was proved in [NT] by using a microlocal calculation of Demailly.

Theorem 1.1. *Let (M, g) be a complete Kähler manifold with Ricci $(g) \leqq -g$. Set $m = \dim M$. Denote by $H^0_{(2)}(M, K_M^{\otimes v})$ the vector space of all square integrable*

(relative to g) holomorphic sections of the v-th tensor power $K_M^{\otimes v}$ of the canonical bundle K_M. Then we have the following estimate:

$$\liminf_{v\to\infty} v^{-m} \dim H^0_{(2)}(M, K_M^{\otimes v}) \geqq \frac{1}{m!}\int_M c_1(K_M)^m \ ,$$

where $c_1(K_M) = -$ Ricci(g) is the Chern-Weil form of the canonical bundle K_M equipped with the metric induced from ω.

1.5. The relationship between L^2-forms and cusp forms

Proposition 1.10. *Let M be a complex manifold of dimension m, D an effective reduced divisor on M, and g a complete Kähler metric on $M - D$ such that Ricci(g) $\leqq -g$. Then the volume form g^m is integrable over any compact subset of M.*

Proof. By Hironaka's resolution of singularities we may assume without loss of generality that D has simple normal crossings. It is sufficient to show that the volume form g^m is integrable locally along D. Take a unit polydisc

$$\Delta^m = \{(z^1, \ldots, z^m) \in \mathbf{C}^m : |z^i| \leqq 1,\ 1 \leqq i \leqq m\}$$

in M such that

1. $\Delta^m \cap D = \{z^1 \ldots z^k = 0\}$, and
2. $\Delta^m \cap (M - D) = (\Delta^*)^k \times \Delta^{m-k}$.

Consider the Poincaré metric on $(\Delta^*)^k \times \Delta^{m-k}$ given explicitly by

$$g_P = \frac{\sqrt{-1}}{2}\sum_{i=1}^{k} \frac{dz^i \otimes d\bar{z}^i}{|z^i| \log^2 |z^i|^2} + \frac{\sqrt{-1}}{2}\sum_{i=k+1}^{m} \frac{dz^i \otimes d\bar{z}^i}{(1 - |z^i|^2)^2} \ .$$

We apply the Schwarz lemma to the inclusion $((\Delta^*)^k \times \Delta^{m-k}, g_P) \subsetneq (M - D, g)$ to conclude that $g^m \leqq$ (constant)g_P^m. In applying the Schwarz lemma we have used the fact that the Ricci curvature of g is bounded from above by a negative constant while that of g_P is bounded from below by a constant. Since the Poincaré volume form is integrable locally along D we are done. QED

Proposition 1.11. *Let M be a complex manifold, D an effective reduced divisor on M, and g a complete Kähler metric on $M - D$ such that Ricci(g) $\leqq -g$. Then*

$$H^0_{(2)}(M - D, K^{\otimes v}_{M-D}) \subset H^0(M, K_M^{\otimes v} \otimes [D]^{\otimes(v-1)}) \ .$$

(Here the left hand side denotes the space of all holomorphic v-canonical forms on $M - D$ which are square integrable relative to g.)

Proof. Let $\psi \in H^0_{(2)}(M - D, K^{\otimes v}_{M-D})$ be given. For any compact subset C of M we have

$$\int_C (\psi \wedge \bar{\psi})^{\frac{1}{v}} = \int_C |\psi|^{\frac{2}{v}} g^m$$

$$\leqq \int_C (|\psi|^2 + 1) g^m < \infty$$

since ψ is square integrable and since g^m is integrable on compact subsets of M by Proposition 1.10. Here we have used the algebraic inequality $X^{\frac{1}{\nu}} \leqq X + 1$ (for $X > 0$, $\nu \geqq 1$). Since the set C can be made arbitrarily large we see that ψ extends meromorphically to M with strictly less than logarithmic poles along D (Proposition 1.6). The proof is complete. QED

1.6. The existence of quasiprojective structure on certain Zariski-open subsets of a compact complex manifold

The following result is due to Mok and Wong [MW], although we present a proof here for the sake of completeness.

Proposition 1.12. *Let M be a compact complex manifold and let $U \subset M$ be a Zariski open subset which is covered by a bounded domain of holomorphy $\tilde{U}$ (in $\mathbf{C}^m$ or more generally spread over a Stein manifold). Then U is biholomorphic to a unique quasiprojective variety of log-general type.*

Remark. We do not require $\tilde{U}$ to be the universal cover of U.

Proof. By [MY] we know that $\tilde{U}$ admits a complete Kähler-Einstein metric; by uniqueness this metric descends to a complete Kähler-Einstein metric $g = -Ricci(g)$ on U. By Proposition 1.11 we find that $H^0_{(2)}(U, K_U^{\otimes \nu}) \subset \{\text{cusp forms on } U\}$. Because g has uniformly negative Ricci curvature, we can use L^2-estimates of $\bar{\partial}$ to show that

$$\bigcup_{\nu=1}^{\infty} H^0_{(2)}(U, K_U^{\otimes \nu})$$

separates the points of U and forms local coordinates on U. Hence the cusp forms on U separate the points of U and form local coordinates on U.

Consider now the holomorphic line bundle L on M defined by $L = K_M \otimes [D]$ where D is a reduced effective divisor on M whose support is precisely $M - U$. (Since U is uniformized by a Stein manifold it follows that $M - U$ has pure codimension 1 in M.) Then

$$\bigcup_{\nu=1}^{\infty} H^0(M, L^{\otimes \nu})$$

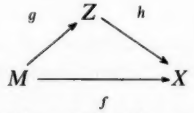
separates the points of U and forms local coordinates on U. Because M is compact we can employ Noetherian induction (e.g. induction on the dimension of the base locus) to find ν_0 such that

$$H^0(M, L^{\otimes \nu_0})$$

separates the points of U and forms local holomorphic coordinates on U. This gives rise to a meromorphic map

$$M \dashrightarrow \mathbf{P}$$

into some (finite dimensional) projective space **P**. After bimeromorphically modifying M outside of U we can assume that this map is in fact holomorphic. As a result we obtain a holomorphic map $f: M \to X$ whose restriction to U is injective and immersive, where X is an irreducible complex projective variety of the same dimension as M. Moreover, we may assume that X is normal (since M is). Consider now the Stein factorization



where g has connected fibers, h has finite fibers, and Z is a normal connected compact complex space. Since the restriction of f to U is one-to-one, we conclude that h has degree one and hence that f has connected fibers. It follows that X is the *disjoint* union of $f(U)$ and $f(M - U)$. By Remmert's proper mapping theorem, $f(M - U)$ is a subvariety of X and we find that $f(U)$ is Zariski open in X. Thus f gives a biholomorphism of U onto the quasiprojective variety $f(U)$. This proves the existence of quasiprojective structure on U. This quasiprojective structure is automatically of log-general type since the cusp forms on U form local coordinates. Uniqueness follows from Proposition 1.9. QED

1.7. The open embedding

About 25 years ago Rossi [Ros] proved that any strictly pseudoconcave complex manifold of dimension at least three can be embedded as an open (in the classical topology) subset of a compact complex manifold. (In dimension two the result is false.) Variations on Rossi's theme which deal with open embeddings into projective algebraic manifolds (and which also handle the two-dimensional case) were later considered by Andreotti-Tomassini [AS] and Andreotti-Siu [AS]. Below we present Rossi's result in a suitable form, while postponing to subsequent sections the much more difficult problem of determining when the open embedding is actually Zariski-open.

Theorem 1.2. *Let M be a connected complex manifold of dimension $m \geqq 3$. Assume that there exists a continuous proper map $\varphi: M \to [0, \infty)$ such that $-\varphi$ is strictly plurisubharmonic outside of a compact set. Then M is (biholomorphic to) an open subset of a compact complex manifold $\bar{M}$. Moreover, the following statements are true:*

1. *The set $\bar{M} - M$ has measure 0 in M.*
2. *Every meromorphic function or tensor on M extends meromorphically to $\bar{M}$.*
3. *Every subvariety of M without 0 or 1-dimensional components extends to a subvariety of $\bar{M}$.*

Proof. By Richberg's smoothing lemma [Ric] or [FS, Theorem 8.2 on p. 38] we can assume that φ is smooth. Then the existence of $\bar{M}$ is proved in [Ros, AS]. Statement (1) follows from the fact that $\bar{M} - M$ is a pluripolar set (extend φ to all

of $\bar{M}$ by defining it to be ∞ on $\bar{M} - M$). Statements (2) and (3) follow from the strict pseudoconcavity of M and may be regarded as Hartogs-type extension theorems; for (2) see [KR, Theorem 7.5 on p. 470] and for (3) see [GR, Corollary 7 on p. 229]. QED

2. Zariski-local Kähler-Einstein geometry

This section forms the technical heart of the paper. Its goal is to give a criterion for certain (classically) open subsets of compact complex manifolds to be Zariski-open; the main results are stated in Theorems 2.1 and 2.2 below. The proofs follow the general line of reasoning which was introduced in [NT] and in particular make use of the following three deep results: (1) The existence of Kähler-Einstein metrics [Kob, MY], (2) The L^2 Riemann-Roch Inequality [NT], and (3) The Zariski-local uniformization of algebraic varieties by bounded domains of holomorphy [Gri]. In hope of avoiding potential confusion in the proofs we shall be careful to distinguish between subvarieties of pure codimension 1 and divisors.

Theorem 2.1. *Let M be a smooth complex projective variety and let $U \subset M$ be an open (in the classical topology) subset. Assume the following:*

1. *The set $M - U$ has measure 0 in M and contains a subvariety $V \subset M$ of pure codimension 1 which is maximum (among all subvarieties of pure codimension 1 which are contained in $M - U$).*
2. *The universal cover of U is Stein.*
3. *For every subvariety $H \subset M$ of pure codimension 1, every integer $v > 0$, and every $\psi \in H^0(U - H, K_U^{\otimes v})$ such that*

$$\int_U (\psi \wedge \bar{\psi})^{\frac{1}{v}} < \infty$$

it follows that ψ extends meromorphically to a v-canonical form on all of M.

4. *For every subvariety $H \subset M$ of pure codimension 1, if $U - H$ admits a complete Kähler-Einstein metric $g = -Ricci(g)$ then it follows that $vol_g(U - H) < \infty$.*

Then $U = M - V$. In particular, U is Zariski-open in M.

Proof. Let $V' \subset M$ be a subvariety such that $M - V'$ is uniformized by a bounded domain in $\mathbf{C}^m$ [Gri]. By applying Hironaka's resolution of singularities if necessary, we can assume without loss of generality that $V \cup V'$ is the support of a divisor D on M with simple normal crossings. Let D' be a smooth ample divisor on M such that

1. The divisor $D + D'$ has simple normal crossings.
2. The divisor $K_M + D + D'$ is ample on M.

Let H = the support of $D + D'$.

Claim 1. $U - H$ is covered by a bounded domain of holomorphy in $\mathbf{C}^m$. (This domain need not be the *universal* cover.)

Proof. Since the universal cover of $U - H$ is Stein (cf. Propositions 1.2 and 1.3), it is sufficient to show that $U - H$ is covered by *some* bounded domain in $\mathbf{C}^m$ (Proposition 1.4). But this is clear since $M - V'$ is.

Claim 2. $U - H$ admits a complete Kähler-Einstein metric $g = -Ricci(g)$ of finite volume.

Proof. Since $U - H$ is covered by a bounded domain of holomorphy, it admits a complete Kähler-Einstein metric $g = -Ricci(g)$ [MY] which must be of finite volume by hypothesis (4).

Claim 3. $M - H$ admits a complete Kähler-Einstein metric $g' = -Ricci(g')$ of volume $(K_M + D + D')^m$.

Proof. This follows from [Kob] since $K_M + D + D'$ is ample and $D + D'$ has simple normal crossings.

Claim 4. $(g')^m \leqq g^m$ on $U - H$.

Proof. This is a consequence of the Schwarz lemma [MY, Ya].

Claim 5.

$$H^0_{(2)}(U - H, K^{\otimes \nu}_{U-H}) \subset H^0(M, K^{\otimes \nu}_M \otimes [D + D']^{\otimes(\nu-1)}) .$$

Proof. For any $\psi \in H^0_{(2)}(U - H, K^{\otimes \nu}_{U-H})$ we have

$$\begin{aligned} \int_U (\psi \wedge \bar{\psi})^{\frac{1}{\nu}} &= \int_U |\psi|^{\frac{2}{\nu}} g^m \\ &\leqq \int_U (|\psi|^2 + 1) g^m \\ &< \infty \end{aligned}$$

since ψ is square-integrable and since $U - H$ has finite g-volume. Here we have used the simple algebraic inequality $X^{\frac{1}{\nu}} \leqq X + 1$ (for $X > 0, \nu \geqq 1$). Hence by assumption (3) we know that ψ extends meromorphically to M. Its poles must lie in H and must be strictly less than logarithmic since

$$\int_M (\psi \wedge \bar{\psi})^{\frac{1}{\nu}} = \int_U (\psi \wedge \bar{\psi})^{\frac{1}{\nu}} < \infty .$$

This proves Claim 5.

Claim 6.

$$\int_U g^m \leqq \int_U (g')^m .$$

Proof. We have

$$\frac{1}{m!}\int_U g^m \leqq \liminf_{\nu\to\infty} \nu^{-m} \dim H^0_{(2)}(U - H, K^{\otimes\nu}_{U-H}) \text{ (by } L^2\text{-Riemann-Roch)}$$

$$\leqq \liminf_{\nu\to\infty} \nu^{-m} \dim H^0(M, K_M^{\otimes\nu} \otimes [D + D']^{\otimes\nu}) \text{ (by Claim 5)}$$

$$= \frac{1}{m!}(K_M + D + D')^m \text{ (by the Riemann-Roch Theorem)}$$

$$= \frac{1}{m!}\int_M (g')^m$$

$$= \frac{1}{m!}\int_U (g')^m \text{ (since } M - U \text{ has measure 0)} .$$

Completion of proof. By Claims 4 and 6 it follows that $g^m = (g')^m$ on $U - H$ and by the Kähler-Einstein property of the two metrics it follows that $g = g'$ on $U - H$. Since g is complete we conclude that $U - H = M - H$. But V' and D' can "move around". More precisely, while choosing V' and D' we had enough freedom to ensure that any pre-assigned point $p \in M$ be disjoint from $V' \cup$ (the support of D'). It follows that $U = M - V$. This completes the proof of Theorem 2.1. QED

Theorem 2.2. *Theorem* 2.1 *remains valid when M is assumed only to be a compact Moishezon manifold.*

Proof. Let $f: M' \to M$ be a bimeromorphic holomorphic map from a smooth complex projective variety M' as in Proposition 1.5 and let $C \subset M$ be the corresponding center. Let $H \subset M$ be an arbitrary subvariety of pure codimension 1 such that $H \supset C$. We apply Theorem 2.1 to $f^{-1}(U - H) \cong U - H$ inside M' to conclude that $f^{-1}(U - H)$ is Zariski-open in M'. It follows from Remmert's proper mapping theorem that $U - H$ is Zariski-open in M. The complement of $U - H$ in M has pure codimension 1 (by Proposition 1.1 since $U - H$ is uniformized by a Stein manifold) and is therefore equal to the maximal subvariety of pure codimension 1 which it contains, namely $H \cup V$. That is,

$$U - H = M - (H \cup V) .$$

It follows in particular that

$$M - U \subset H \cup V .$$

By moving H around (see Proposition 1.5) we conclude that

$$M - U \subset C \cup V ,$$

and by Proposition 1.1 we conclude further that

$$M - U \subset V .$$

Since we already have the reverse inclusion we find that $M - U = V$, as desired. This completes the proof of Theorem 2.2. QED

3. Proof of Theorem 0.1

Let M be as in Theorem 0.1 and let $\bar{M}$ be a connected compact complex manifold which contains M as an open subset, as in Theorem 1.2. By assumption M admits $m = \dim M$ meromorphic functions which are algebraically independent over $\mathbf{C}$; these extend meromorphically to $\bar{M}$ (Theorem 1.2, part 2) and it follows that $\bar{M}$ is Moishezon.

To complete the proof it suffices to show that M is Zariski-open in $\bar{M}$, for then $\bar{M} - M$ can be blown down to finitely many points since M is strictly pseudoconcave.

Let U be a Zariski-open subset of M which is uniformized by a Stein manifold. Since M is the union of such U, it suffices to show that U is Zariski-open in $\bar{M}$. For this we apply Theorem 2.2. All that remains is to verify the conditions of that theorem. We consider each condition in turn.

Verification of the first condition

Part A. We note that $\bar{M} - U$ has measure 0 in $\bar{M}$ because $\bar{M} - M$ does (since it is a pluripolar set) and so does $M - U$.

Part B. We must show that $\bar{M} - U$ contains a maximum subvariety V (of $\bar{M}$) of pure codimension 1. We note that $\bar{M} - M$ contains a maximum subvariety V' of pure codimension 1 since M is strictly pseudoconcave. We next note that $U = M - V''$ where V'' is a subvariety of $\bar{M}$ of pure codimension 1. The reason for this is that $M - U$ has pure codimension 1 in M since U is uniformized by a Stein manifold (see Proposition 1.1) and that every subvariety of M without 0 or 1-dimensional components extends to a subvariety of $\bar{M}$ (see Theorem 1.2, part 3). Finally, we set $V = V' \cup V''$; this does the job.

Verification of the second condition

There is nothing to verify here since the universal cover of U is Stein by assumption.

Verification of the third condition

Let $H \subset \bar{M}$ be a subvariety. If ψ is a cusp form on $M - H$ then ψ extends meromorphically first to M (Proposition 1.6) and finally to $\bar{M}$ (Theorem 1.2, part 2). (Here we have lost no generality in assuming that $U = M$, for U itself is already of the form $M - H$. We apply similar considerations below when we verify the fourth condition.)

Verification of the fourth condition

Let $H \subset \bar{M}$ be a subvariety of pure codimension 1 and suppose that $g = -Ricci(g)$ is a complete Kähler-Einstein metric on $M - H$. We must show

that $\mathrm{vol}_g(M) < \infty$. We have

$$\frac{1}{m!}\int_M g^m \leqq \liminf_{\nu\to\infty} \nu^{-m} \dim H^0_{(2)}(M - H, K^{\otimes\nu}_{M-H}) \text{ (by } L^2\text{-Riemann-Roch)}$$

$$\leqq \liminf_{\nu\to\infty} \nu^{-m} \dim H^0(M, L^{\otimes\nu})$$

(by Proposition 1.11, where $L = K_M \otimes [D]$)

$$< \infty$$

since M is pseudoconcave [And]. This completes the proof of Theorem 0.1.

4. Proof of Theorems 0.2 and 0.3

Proof of Theorem 0.2. The proof of Theorem 0.2 is exactly the same as that of Theorem 0.1 except that for the open embedding $M \subset \bar{M}$ one uses the result of Andreotti-Tomassini [AT, AS] rather than the result of Rossi. This allows one to assume that $\bar{M}$ is projective algebraic.

Proof of Theorem 0.3. Let M be as in Theorem 0.3. First we show that M is Moishezon. Let $U \subset M$ be a Zariski-open subset of M which is uniformized by a bounded domain of holomorphy in $\mathbf{C}^m$. Then there exists a complete Kähler-Einstein metric $g = -\mathit{Ricci}(g)$ on M. Fix any point $p \in U$. Because g has uniformly negative Ricci curvature we can use Hörmander's L^2-estimates of $\bar{\partial}$ to find an integer $\nu > 0$ together with forms

$$\psi_0, \psi_1, \ldots, \psi_m \in H^0_{(2)}(U, K^{\otimes\nu}_U)$$

such that ψ_0 does not vanish at p while $\psi_1, \ldots, \psi_m$ form a system of local holomorphic coordinates centered at p. As these forms are square integrable, they extend meromorphically to M (Proposition 1.11) and we obtain meromorphic functions

$$\frac{\psi_1}{\psi_0}, \ldots, \frac{\psi_m}{\psi_0}$$

on M which form a system of local holomorphic coordinates at p. It follows that M is Moishezon. Now by Theorem 0.1 we can assume that M is a Zariski-open subset of a compact complex manifold $\bar{M}$. Let $\mathscr{C}$ be the set of all Zariski-open subsets of M which are covered by bounded domains of holomorphy in $\mathbf{C}^m$ (we do not require these to be the *universal* covers). By assumption $\mathscr{C}$ is a covering of M.

Claim. $\mathscr{C}$ is closed under finite intersection.

Proof of claim. Suppose that $U, U' \in \mathscr{C}$; we must show that $U \cap U'$ is covered by a bounded domain of holomorphy in $\mathbf{C}^m$. We note that the universal cover of $U \cap U'$ is Stein because that of U is Stein and because $M - U'$ has pure codimension 1 (cf. Propositions 1.1, 1.2, and 1.3). It is therefore sufficient (by Proposition

1.4) to show that $U \cap U'$ is covered by *some* bounded domain in $\mathbf{C}^m$. But this is clear since U is.

Note that every $U \in \mathscr{C}$ is Zariski-open in $\bar{M}$ (Theorem 1.2, part 3) and hence admits the structure of a unique quasiprojective variety of log-general type (Proposition 1.12). Since these quasiprojective structures respect inclusions (Proposition 1.8), we can endow M with the structure of an abstract algebraic variety by gluing. This completes the proof of Theorem 0.3.

5. A counterexample

In this section we present an example of a complex manifold M which satisfies all of the conditions of Theorem 0.1 except for condition (3) and which is not of finite topological type. It follows that condition (3) is logically independent of the other two conditions.

Fix $m \geqq 3$ and consider the subset $S \subset \mathbf{C}^m$ given by

$$S = \left\{ \left(\frac{1}{\nu}, 0, \ldots, 0 \right) | 1 \leqq \nu \leqq \infty \right\} .$$

Then S is a countable compact set whose only accumulation point is at the origin. Consider now the function η: $\mathbf{C}^m \to [-\infty, 0)$ given by

$$\eta(z) = |z|^2 + \log|z| + \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \log|z - z_\nu|$$

where $z_\nu = \left(\frac{1}{\nu}, 0, \ldots, 0 \right) \in S$. It is not difficult to check that η is continuous (by using the Weierstrass M-test, for example), strictly plurisubharmonic ("strictly" because of the term $|z|^2$), and that $S = \eta^{-1}(-\infty)$. (It follows that, in particular, S is a pluripolar set.)

Now consider $\mathbf{C}^m$ as an open subset of $\mathbf{P}^m$ in the usual way and let $M = \mathbf{P}^m - S$. Then M satisfies condition (1) of Theorem 0.1 (take $\varphi = -\eta$ near S) as well as condition (2) (obvious). However, M is not of finite topological type because S is infinite. Hence M cannot satisfy condition (3).

We remark that this example can be easily generalized: replace $\mathbf{P}^m$ by an arbitrary smooth projective variety and S by any countably infinite closed subset possessing only finitely many accumulation points.

References

[And] Andreotti, A.: Théorèmes de dépendance algébrique sur les espaces pseudoconcaves. Bull. Soc. Math. France **91**, 1–38 (1963)

[AG] Andreotti, A., Grauert, H.: Algebraische Körper von automorphen Funktionen. Nachr. Akad. Wiss. Gött., Math.-Phys. Kl. II., 39–48 (1961)

[AS] Andreotti, A., Siu, Y.-T.: Projective embedding of pseudoconcave spaces. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. Vol. XXIV, Fasc. II, 231–278 (1970)

[AT] Andreotti, A., Tomassini, G.: Some remarks on pseudoconcave manifolds. Essays in Topology and Related Topics, dedicated to G. deRham. Haefliger, A., Narasimhan, R. (eds.). Berlin-Heidelberg-New York: Springer 1970, pp. 85–104

[Art] Artin, M.: Algebraization of formal moduli: II. Existence of modifications. Ann. Math. **91**, 89–135 (1970)

[AMRT] Ash, A., Mumford, D., Rapoport, M., Tai, Y.: Smooth compactifications of locally symmetric varieties. Brookline: Math. Sci. Press 1975

[BB] Baily, W.L., Jr., Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains. Ann. Math. **84**, 422–528 (1966)

[FS] Fornæss, J., Stensønes, B.: Lectures on counterexamples in several complex variables (Math. Notes, Vol. 33): Princeton: Princeton Univ. Press 1987

[Gri] Griffiths, P.: Complex-analytic properties of certain Zariski dense open sets on algebraic varieties. Ann. Math. **94**, 21–51 (1971)

[GR] Gunning, R., Rossi, H.: Analytic functions of several complex variables. Englewood Cliffs, New Jersey: Prentice-Hall 1965

[Kob] Kobayashi, R.: Existence of Kähler-Einstein metrics on open algebraic manifolds. Osaka J. Math. **21**, 399–418 (1984)

[KR] Kohn, J., Rossi, H.: On the extension of holomorphic functions from the boundary of a complex manifold. Ann. Math. **81**, 451–472 (1965)

[M1] Mok, N.: An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties. Bull. Soc. Math. France **122**, 197–258 (1984)

[M2] Mok, N.: Compactification of complete Kähler surfaces of finite volume satisfying certain curvature conditions. Preprint

[MW] Mok, N., Wong, B.: Characterization of bounded domains covering Zariski dense subsets of compact complex spaces. Am. J. Math. **105**, 1481–1487 (1983)

[MY] Mok, N., Yau, S.-T.: Completeness of the Kähler-Einstein metric on bounded Riemann domains and the characterisation of domains of holomorphy by curvature conditions. Proc. Symp. Pure Math. **39**, 41–59 (1983)

[MZ] Mok, N., Zhong, J.: Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature. Preprint

[NT] Nadel, A., Tsuji, H.: Compactification of complete Kähler manifolds of negative Ricci curvature. J. Differ. Geom. **28**, 503–512 (1988)

[Ric] Richberg, R.: Stetige streng pseudokonvex Funktionen. Math. Ann. **175**, 251–286 (1968)

[Ros] Rossi, H.: Attaching analytic spaces to an analytic space along a pseudoconcave boundary. Proc. Conf. Complex Analysis (Minneapolis 1964). Berlin-Heidelberg-New York: Springer 1965, pp. 242–256

[Sak] Sakai, F.: Kodaira dimension of complement of divisors. Complex Analysis and Algebraic Geometry; a collection of papers dedicated to K. Kodaira. Tokyo: Iwanami 1977

[Sat] Satake, I.: On compactifications of the quotient spaces for arithmetically defined discontinuous groups. Ann. Math. **72**, 555–580 (1960)

[Siu] Siu, Y.-T.: Pseudoconvexity and the problem of Levi. Bull. Am. Math. Soc. **84**, 481–512 (1978)

[SY] Siu, Y.-T., Yau, S.-T.: Compactification of negatively curved complete Kähler manifolds of finite volume. Seminar on Diff. Geom., Yau, S.-T. (ed.). Ann. Math. Stud. **102**, 363–380 (1980)

[Ya] Yau, S.-T.: A general Schwarz lemma for Kähler manifolds. Am. J. Math. **100**, 197–203 (1978)

[Ye] Yeung, S.-K.: Compactification of complete Kähler surfaces with negative Ricci curvature. Invent. Math. **99**, 145–163 (1990)

[Ye'] Yeung, S.-K.: Compactification of complete Kähler manifolds with negative Ricci curvature. M.I.T. preprint (1989)

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Weak positivity and the stability of certain Hilbert points, II

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In the first part of this paper [14] we proved that all points x of the Hilbert scheme of ν-canonically embedded compact complex manifolds are stable with respect to the usual group action and with respect to some invertible sheaf, provided that x is a smooth point of H_{red}. As explained there, the only reason that we had to restrict ourselves to smooth points of H_{red}, was our unability to give an affirmative answer to "problem 1.10" in [14]. In the second part, we will fill this gap (see 2.9) and thereby finish the proof of

Theorem 0.1. *There exists a quasi-projective coarse moduli space M_h for complex compact canonically polarized manifolds with Hilbert polynomial h.*

In fact, as in [14] our approach applies to a larger class of moduli problems:

0.2. Let $h(T)$ be a given polynomial of degree n and $\mathcal{M}'_h$ the moduli functor of complex projective normal irreducible varieties, with at most rational Gorenstein singularities and with an ample canonical sheaf ω_F satisfying $h(\nu) = \chi(F, \omega_F^\nu)$. If $n \leqq 2$ we take $\mathcal{M}_h = \mathcal{M}'_h$ and for $n = 3$ we take $\mathcal{M}_h$ to be the subfunctor of $\mathcal{M}'_h$ of families $f: X \to Y$ with $h^0(F, \omega_F) > 0$ for all singular fibers F of f. For $n > 3$ we take $\mathcal{M}_h$ to be the subfunctor of $\mathcal{M}'_h$ of smooth families. In any case $\mathcal{M}_h$ will be a separated and bounded moduli functor (see [8], §2 for those notations) and, in fact, any subfunctor $\mathcal{M}_h$ of $\mathcal{M}'_h$ having both properties can be choosen (see §6).

Theorem 0.3. *For all the moduli functor $\mathcal{M}_h$ considered in* 0.2 *there exists a quasi-projective coarse moduli scheme M_h.*

Our result reproves D. Gieseker's theorem [4] on the existence of quasi-projective moduli spaces for surfaces of general type. However, as we point out in 6.4, the ample sheaf obtained by D. Gieseker is "better" than the one constructed here.

Narasimhan-Simha, Tankeev, Popp, Mumford-Fogarty and Kollár (see [14] for the references) have shown the existence of M_h as an analytic space. If one does not like D. Mumford's geometric invariant theory [10] one can use instead [8], 2.7, together with 5.2 (or 1.18 of [14]) in order to construct an ample sheaf on the

analytic moduli space directly. However, this only seems to work at present, if the non normal locus of M_h is compact. Otherwise this method will show only that the normalization of M_h is quasi-projective (see 5.9).

The methods used to answer "problem 1.10" of [14] in 2.9, are similar to those, already employed in §3 of [14], except one, an unpublished theorem on the extension of certain locally free sheaves to compactifications, due to Ofer Gabber (see 1.4 for the exact statement). After he told me about his result and the spirit of his proof, I was able to cook up my own, slightly different and more clumsy version (see 1.6) and its proof.

Those "Extension Theorems" together with a statement on "extensions after finite covers" in 1.7 are discussed in §1. In §2 we recall and extend the definition of weakly positive sheaves and reformulate "problem 1.10" of [14]. So we have to show, that for certain morphisms $f_0: X_0 \to Y_0$ the sheaves $f_{0*}\omega^{\nu}_{X_0/Y_0}$ are weakly positive. The usual arguments (see [9], [12] and [14]) are used in §4 to reduce this problem to the case $\nu = 1$.

"Weak positivity" is a positivity notation which uses the existence of sections (see 2.2), and the only reasonable methods to construct sections are using either ampleness criteria or vanishing theorems. Since both are only valid on compact manifolds or schemes, it is not too surprising that at some point we have to consider compactifications Y of Y_0 and extensions of $f_{0*}\omega^{\nu}_{X_0/Y_0}$ to coherent sheaves on Y. This is done in §3 using the "Extension theorem 1.7" and the "Nilpotent Orbit Theorem" of W. Schmid [11]. Unfortunately there are some complications due to the fact that we do not only want to consider moduli of non singular varieties. Those force us to include the quite technical constructions in the second half of §3.

Once an affirmative answer to "problem 1.10" is established we just have to apply the results of [14] to obtain theorem 0.3. Nevertheless we discuss in §6 the necessary assumptions a moduli functor $\mathcal{M}_h$ has to satisfy in order to get a quasi-projective moduli scheme, by using our method. Finally we recall which parts of [14] and this paper are necessary to obtain theorem 0.3 in different cases, and we make a few remarks on possible generalizations.

In § 5 we just recall the applications 2.9 has for fiber spaces. We could not resist however to formulate the ampleness criterion which, as J. Kollár pointed out, was more or less proven in § 2 and 4 of [14] but not stated there (see also [8], § 3).

This paper was written during a sabbatical term which I used to exploit the hospitality of the I.H.E.S. at Bures sur Yvette.

I thank Hélène Esnault for her help and comments during the preparation of this paper.
Special thanks I owe to Ofer Gabber. Without him, telling me about his extension theorem, this paper would not have been written.

Conventions

We try to use the usual notations of algebraic geometry (as for example in R. Hartshorne's book on this subject) and those from higher dimensional birational geometry (see [9] and [14]).

However, all varieties and manifolds are supposed to be defined over the field $\mathbb{C}$ of complex numbers, and the word "*scheme*" is used for "schemes, separated and of finite type over $\mathbb{C}$". *Locally free sheaves* on a scheme are supposed to be of finite rank.

We call $i: W_0 \to W$ a *compactification* if W is a proper scheme and W_0 an open dense subscheme. If W_0 is reduced (non singular or quasi-projective) we assume that W is reduced (non singular or projective) as well.

A morphism $\delta: W' \to W$ is called *birational* if there is an open dense subscheme W_0 of W such that $W'_0 = \delta^{-1}(W_0)$ is dense in W' and $\delta|_{W'_0}: W'_0 \to W_0$ an isomorphism. The *center* of δ is the complement of the largest W_0 with this property. An example is the blowing up of an ideal sheaf J on W, where the center is the closed subscheme where J is not invertible.

We will call δ a *desingularization*, if δ is a projective morphism and W' non singular.

A morphism $\delta: W' \to W$ will be called *finite and dominant*, if it is finite and if each component of W' is dominant over some component of W. In this case, we have an induced diagram

$$\begin{array}{ccc} \tilde{W}' & \xrightarrow{\tilde{\delta}} & \tilde{W} \\ \downarrow & & \downarrow \tau \\ W' & \xrightarrow{\delta} & W \end{array}$$

where $\tilde{W}$ and $\tilde{W}'$ are the normalizations. We have an inclusion

$$\delta_* \mathcal{O}_{W'} \to \tau_* \tilde{\delta}_* \mathcal{O}_{\tilde{W}}$$

and a trace map $\tau_* \tilde{\delta}_* \mathcal{O}_{\tilde{W}} \to \tau_* \mathcal{O}_{\tilde{W}}$. The induced map $\delta_* \mathcal{O}_{W'} \to \tau_* \mathcal{O}_{\tilde{W}}$ will be called the *trace map* of δ.

Finally, a *point* $x \in W$ is always supposed to be a closed point, if not stated otherwise.

If X is a Cohen Macauley scheme, ω_X will denote the *dualizing sheaf*. If $f: X \to Y$ is a morphism of schemes, Y Gorenstein and X Cohen Macauley, then we write $\omega_{X/Y} = \omega_X \otimes f^* \omega_Y^{-1}$.

If $f: X \to Y$ is flat and Cohen Macauley or Gorenstein, then $\omega_{X/Y}$ denotes the *relative dualizing sheaf*.

If X is a non singular scheme we call D a *normal crossing divisor* if D is a divisor on X, $X - D_{\text{red}}$ open and dense in X and if the components of D are non singular divisors intersecting transversally.

§ 1. Extensions of locally free sheaves to compactifications

1.1. Let us consider a reduced scheme W_0 and a locally free sheaf $\mathcal{F}_0$ on W_0. Assume that there is a desingularization $\delta_0: W'_0 \to W_0$, a non singular compactification $i': W'_0 \to W'$ and a locally free sheaf $\mathcal{F}'$ on W' with $i'^* \mathcal{F}' \cong \delta_0^* \mathcal{F}_0$.

We want to study conditions which imply that $\mathcal{F}'$ comes from some compactification W of W_0. More precisely we want to answer

Problem 1.2. Does there exist a compactification $i: W_0 \to W$ and a locally free sheaf $\mathscr{F}$ on W such that for all commutative diagrams of birational morphisms

$$\begin{array}{ccccc} W'' & \xrightarrow{\delta'} & W' & \xleftarrow{i'} & W_0' \\ {\scriptstyle\delta}\downarrow & & & & \downarrow{\scriptstyle\delta_0} \\ W & & \underset{i}{\leftarrow} & & W_0 \end{array}$$

with W'' non singular, one has $\delta^*\mathscr{F} = \delta'^*\mathscr{F}'$?

Here we use " $=$ " to indicate an isomorphism of sheaves (unique if W'' is proper) which coincides over $\delta'^{-1}(W_0') \cap \delta^{-1}(W_0)$ with the one given in 1.1.

Obviously the answer is no. In fact, the following condition is necessary:

Condition 1.3. Let C be a non singular curve, C_0 an open dense subcurve and $\eta_0: C_0 \to W_0$ a morphism. Then there exists a locally free sheaf $\mathscr{G}$ on C with $\eta_0^*\mathscr{F}_0 \cong \mathscr{G}|_{C_0}$, such that for all diagrams

$$\begin{array}{ccccc} C' & \xrightarrow{\eta'} & W' & \xleftarrow{i'} & W_0' \\ {\scriptstyle\tau}\downarrow & & & & \downarrow{\scriptstyle\delta_0} \\ C & \leftarrow & C_0 & \underset{\eta_0}{\rightarrow} & W_0 \end{array}$$

with τ finite, $\eta'(\tau^{-1}C_0) \subseteq W_0'$ and $\eta_0 \circ \tau|_{\tau^{-1}(C_0)} = \delta_0 \circ \eta'|_{\tau^{-1}(C_0)}$, one has $\tau^*\mathscr{G} = \eta'^*\mathscr{F}'$.

O. Gabber studied the locally ringed space $W^\wedge$ obtained by taking the limit over all compactifications W of W_0. Then he showed that 1.3 implies that one can solve the extension problem locally for $W^\wedge$. Those solutions, however, as he can prove, must come from a locally free sheaf $\mathscr{F}$ defined on a scheme W which compactifies W_0. Thereby he obtains:

Theorem 1.4 (O. Gabber). *Problem* 1.2 *has an affirmative answer if and only if condition* 1.3 *is satisfied.*

For the purpose of this paper we will formulate (in 1.6) and prove a slightly different version of this "extension theorem". We replace 1.3 by asking for the existence of compatible extensions of $\mathscr{F}_0$ to compactifications of non singular strata of W_0. It will be quite obvious that 1.6 is weaker than O. Gabber's theorem. However, as he pointed out, it is likely that both are equivalent.

The reader should have the following type of example in mind: assume that for all desingularizations $\eta_0: Z_0 \to S_0$ of subschemes S_0 of W_0 the sheaf $\eta_0^*\mathscr{F}_0$ has a "natural" integrable connection. Then, if $Z_0 \to Z$ is a compactification of Z and $Z - Z_0$ a normal crossing divisor, $\eta_0^*\mathscr{F}_0$ has a canonical extension as a locally free sheaf $\mathscr{F}_Z$ on Z (see Deligne [2]). If all the monodromies at the boundary are unipotent, the canonical extension will be compatible with pullbacks. Under this assumption we want to extend $\mathscr{F}_0$ to some locally free sheaf $\mathscr{F}$ which gives the canonical extensions on all possible Z.

In general, the condition that the monodromies are unipotent is too stong. Hence, to include the case of quasi-unipotent monodromies, one could also consider generically finite morphisms $\eta_0: Z_0 \to S_0$ and ask for unipotent monodromies, provided the degree of η_0 is large enough. May be, with this type of example in mind, the following definition looks reasonable:

Definition 1.5. Let Y_0 be a reduced scheme.

a) A category $\mathscr{Z}$ of *compactifying tripels* for Y_0 is a category of tripels $\underline{Z} = (Z, Z_0, \eta_0)$, where Z is a non singular proper scheme, Z_0 the complement of a normal crossing divisor in Z and $\eta_0: Z_0 \to Y_0$ a proper morphism. The morphisms

$$\tau: \underline{Z} = (Z, Z_0, \eta_0) \to \underline{Z}' = (Z', Z_0', \eta_0')$$

in $\mathscr{Z}$ are supposed to be all morphisms $\tau: Z \to Z'$ with $\tau(Z_0) \subseteq Z_0'$ and $\eta_0' \circ \tau|_{Z_0} = \eta_0$.

b) We call a category of compactifying tripels *complete*, if for all $\underline{Z} = (Z, Z_0, \eta_0)$ in $\mathscr{Z}$ and for all proper morphisms $\tau: Z' \to Z$, the tripel

$$(Z', Z_0' = \tau^{-1}(Z_0), \eta_0' = \eta_0 \circ \tau|_{Z_0'})$$

belongs to $\mathscr{Z}$ if and only if Z' is non singular and $Z' - Z_0'$ a normal crossing divisor.

c) We say that $\mathscr{Z}$ *covers* Y_0, if $\mathscr{Z}$ is complete and if there are finitely many $\underline{Z}^{(i)} = (Z^{(i)}, Z_0^{(i)}, \eta_0^{(i)}) \in \mathscr{Z}$, let's say for $i = 1, \ldots, k$, such that

i) There is an open dense non singular subscheme $S^{(i)}$ of $\eta_0^{(i)}(Z_0^{(i)})$ such that $\eta_0^{(i)}|_{\eta_0^{(i)-1}(S^{(i)})}: \eta_0^{(i)-1}(S^{(i)}) \to S^{(i)}$ is finite.
ii) $Y_0 = \bigcup_{i=1}^{k} S^{(i)}$.

d) We say that $\mathscr{Z}$ *covers* Y_0 *birationally*, if we can choose $\underline{Z}^{(1)}, \ldots, \underline{Z}^{(k)}$ such that in addition to i) and ii) the morphisms $\eta_0^{(i)}|_{\eta_0^{(i)-1}(S^{(i)})}$ are isomorphisms (and hence $\eta_0^{(i)}$ is a desingularization).

e) Let $\tau_0: W_0 \to Y_0$ be a proper morphism and $i: W_0 \to W$ a compactification. We write $\mathscr{Z}_W$ for the full subcategory of $\mathscr{Z}$ consisting of all $\underline{Z} = (Z, Z_0, \eta_0) \in \mathscr{Z}$ for which $\eta_0: Z_0 \to Y_0$ factors like $Z_0 \xrightarrow{\eta_0'} W_0 \xrightarrow{\tau_0} Y_0$ and η_0' is the restriction of some morphism $\eta': Z \to W$.

f) Let $\mathscr{F}_0$ be a locally free sheaf on Y_0. An *extension* $\mathscr{G}$ *of* $\mathscr{F}_0$ *to* $\mathscr{Z}$ consists of a sheaf $\mathscr{G}_{\underline{Z}}$ on Z for all $\underline{Z} = (Z, Z_0, \eta_0) \in \mathscr{Z}$ with $\eta_0^* \mathscr{F}_0 = \mathscr{G}_{\underline{Z}}|_{Z_0}$ such that for all morphisms $\tau: \underline{Z} \to \underline{Z}'$ in $\mathscr{Z}$ one has $\tau^* \mathscr{G}_{\underline{Z}'} = \mathscr{G}_{\underline{Z}}$.

g) Using the notation from e), let $\mathscr{F}$ be a locally free sheaf on W with $i^* \mathscr{F} = \tau_0^* \mathscr{F}_0$. Let $\mathscr{G}$ be an extension of $\mathscr{F}_0$ to $\mathscr{Z}$. Then we say that $\mathscr{F}$ is *induced* by $\mathscr{G}$ if for all $\underline{Z} \in \mathscr{Z}_W$ and all prolongations $\eta': Z \to W$ of $\eta_0': Z_0 \to W_0$ one has $\eta'^* \mathscr{F} = \mathscr{G}_{\underline{Z}}$.

Again some remark about " = " is necessary: in f) and g) it means that we fix an isomorphism $\eta_0^* \mathscr{F}_0 \cong \mathscr{G}_{\underline{Z}}|_{Z_0}$ and $i^* \mathscr{F} \cong \tau_0^* \mathscr{F}_0$. The equality $\eta'^* \mathscr{F} = \mathscr{G}_{\underline{Z}}$ means that

$\eta_0'^* i^* \mathscr{F} \cong \eta_0^* \mathscr{F}_0 \cong \mathscr{G}_{\underline{Z}}|_{Z_0}$ is the restriction of $\eta'^* \mathscr{F} \cong \mathscr{G}_{\underline{Z}}$. Remark that, since Z is non singular and Z_0 dense in Z, the isomorphism $\eta'^* \mathscr{F} \cong \mathscr{G}_{\underline{Z}}$ is uniquely determined by its restriction to Z_0.

Theorem 1.6. *Let W_0 be a reduced scheme, and $\mathscr{F}_0$ be a locally free sheaf on W_0. Assume that there exists a category $\mathscr{Z}$ of compactifying triples and an extension $\mathscr{G}$ of $\mathscr{F}_0$ to $\mathscr{Z}$. If $\mathscr{Z}$ covers W_0 birationally, then there exists a compactification $i: W_0 \to W$ and a locally free sheaf $\mathscr{F}$ on W such that $i^* \mathscr{F} = \mathscr{F}_0$ and such that $\mathscr{F}$ is induced by $\mathscr{G}$.*

Again, the assumptions made in 1.6 are obviously necessary: if $\mathscr{F}$ and W are given we can take $\mathscr{Z}$ to be the category of all compactifying tripels and define $\mathscr{G}$ on $\mathscr{Z}_W$ by $\mathscr{G}_{\underline{Z}} = \eta^* \mathscr{F}$ (using the notations from 1.5.g). Before proving 1.6 let us state as consequence:

Theorem 1.7. *Let Y_0 be a reduced scheme and $\mathscr{F}_0$ a locally free sheaf on Y_0. Assume that there exists some category $\mathscr{Z}$ of compactifying tripels for Y_0 and an extension $\mathscr{G}$ of $\mathscr{F}_0$ to $\mathscr{Z}$. If $\mathscr{Z}$ covers Y_0 then we can obtain a finite dominant morphism $\tau_0: W_0 \to Y_0$ and a compactification $i: W_0 \to W$ and a locally free sheaf $\mathscr{F}$ on W such that*

a) *The trace map from $\tau_{0*} \mathscr{O}_{W_0}$ to the integral closure $\mathscr{O}_{\tilde{Y}_0}$ of $\mathscr{O}_{Y_0}$ factors over $\mathscr{O}_{Y_0}$.*
b) *$i^* \mathscr{F} = \tau_0^* \mathscr{F}_0$ and $\mathscr{F}$ is induced by $\mathscr{G}$.*

Since the assumption in 1.7 are the same as in 1.6, except of the missing word "birationally", 1.7 is a consequence of 1.6 and of the following lemma:

Lemma 1.8. *Let Y_0 be a reduced scheme and $\mathscr{Z}$ a category of compactifying tripels which covers Y_0. Then there exists a finite dominant morphism $\tau_0: W_0 \to Y_0$ such that:*

a) *The trace map from $\tau_{0*} \mathscr{O}_{W_0}$ to the integral closure $\mathscr{O}_{\tilde{Y}_0}$ of $\mathscr{O}_{Y_0}$ factors over $\mathscr{O}_{Y_0}$.*
b) *The category $\mathscr{Z}'$ of all compactifying tripels $\underline{Z} = (Z, Z_0, \eta_0)$ of W_0 with $(Z, Z_0, \tau_0 \circ \eta_0) \in \mathscr{Z}$ covers W_0 birationally.*

We will prove 1.8 at the end of this section. To prove 1.6 we need the following lemma. We remind that all schemes are of finite type over $\mathbb{C}$ and separated.

Lemma 1.9. *Let W and W' be reduced schemes, S a closed subscheme of W, S_0 an open dense subscheme of S and $\delta: W' \to W$ a desingularization with center in S such that $\delta^{-1}(S)$ is a divisor. Let T be the closure of $\delta^{-1}(S_0)$ in W' and E an effective divisor on W' with $E \cap T = \varnothing$. Then we can find a diagram of birational morphisms*

$$\begin{array}{ccc} V' & \xrightarrow{\varepsilon} & W' \\ \downarrow \rho & & \downarrow \delta \\ V & \xrightarrow{\sigma} & W \end{array}.$$

and a Cartier divisor D on V with:

i) *The centers of σ and ε lie in $S - S_0$ and $\delta^{-1}(S - S_0)$ respectively.*
ii) *If S' is the proper transform of S under σ, then ρ is a desingularization with center in S'.*
iii) *$D \cap S' = \varnothing$ and $\delta^* D = \varepsilon^* E$.*

Proof. If f is any morphism we write $f'(\ )$ instead of $f^*(\ )$/torsion. Let us denote $S - S_0$ by C. By assumption $\delta' I_s$ is invertible, where I_s is the ideal sheaf of S, and for some effective divisor Δ with $\delta(\Delta) \subset C$ we can write $\delta' I_s = \mathcal{O}_{W'}(-T-\Delta)$. We have $\delta(E) \cap S \subset C$ and, since W is non singular outside of S, $\delta(E)|_{W-C}$ is a Cartier divisor.

We choose I to be an ideal sheaf such that I coincides with $\mathcal{O}_W(-\delta(E))$ on $W - C$ and such that $\delta' I \subset \mathcal{O}_{W'}(-E-\Delta)$. For example, for $m \gg 0$ we can choose $I = \mathcal{O}_W(-\delta(E)) \cap I_C^m$. The assumptions of 1.7 are compatible with blowing up W', as long as the center lies in $\delta^{-1}(S - S_0)$. Therefore we may assume that $\delta' I = \mathcal{O}_{W'}(-\Sigma)$ for some effective divisor Σ supported in $E + \delta^{-1}(C)$. Consider the ideal sheaf $J = (I_s \cup I) \cdot \mathcal{O}_W$. We have an inclusion $I \subset J$ and $\operatorname{Im}(J \to \mathcal{O}_s) = \operatorname{Im}(I \to \mathcal{O}_s)$.

$\delta' J$ is invertible outside of $\delta^{-1}(C)$ and we can choose $\varepsilon: V' \to W'$ to be any blowing up with center in $\delta^{-1}(C)$ such that V' is non singular and $\varepsilon'\delta' J$ invertible. For some effective divisor Γ on V' supported in $\varepsilon^* E + \varepsilon^{-1}\delta^{-1}(C)$ we can write $\varepsilon'\delta' J = \mathcal{O}_{V'}(-\varepsilon^*\Sigma + \Gamma)$.

If T' denotes the proper transform of T in V' we still have

$$\operatorname{Im}(\varepsilon'\delta' J \to \mathcal{O}_{T'}) = \operatorname{Im}(\varepsilon^*\delta' I \to \mathcal{O}_{T'})$$

and hence Γ will not meet T'. Since $\varepsilon^*\delta' I_s$ is also contained in $\varepsilon'\delta' J$ we have $\varepsilon^*\Delta + \varepsilon^* T \geqq \varepsilon^*\Sigma - \Gamma$ and, by the choice of I, $\varepsilon^*\Sigma \geqq \varepsilon^* E + \varepsilon^*\Delta$. Then $\Gamma + \varepsilon^* T \geqq \varepsilon^* E$ and, since $E \cap T = \varnothing$ we find $\Gamma \geqq \varepsilon^* E$.

J and I are both invertible outside of C and we can choose $\sigma: V \to W$ to be the successive blowing up of J and I. By the universal property of blowing ups we find $\varphi: V' \to V$. Blowing up a little bit more we may assume 1.9, ii to hold true. We have an inclusion $\sigma' I \to \sigma' J$ and, since both are invertible, $\sigma' J = \mathcal{O}_V(D') \otimes \sigma' I$ for an effective Cartier divisor D'. By construction $\rho^* D' = \Gamma \geqq \varepsilon^* E$. Since $\rho^* D'$ does not meet the closure T' of $\rho^{-1}(\sigma^{-1}(S_0))$, D' can not meet the closure S' of $\sigma^{-1}(S_0)$. Therefore D' does not meet the singular locus of V and we can choose for D some subdivisor of D'.

Proof of 1.6. We proceed by induction on $\dim(W_0)$. If $\dim(W_0) = 0$ there is nothing to show. Let $\dim(W_0) = n$. Since $\mathfrak{Z}$ covers W_0 birationally, it must contain some desingularization of each component of W_0. Of course, we may assume that $\mathfrak{Z}$ is closed under finite disjoint unions and therefore we have:

a) There is a compactification W of W_0 and a desingularization $\delta: W' \to W$ such that $\underline{W}' = (W', W_0' = \delta^{-1}(W_0), \delta_0 = \delta|_{W_0'}) \in \mathfrak{Z}$. Let us write $\mathcal{G} = \mathcal{G}_{\underline{W}'}$.

If we choose the center S of δ as small as possible, we may assume that $S_0 = S \cap W_0$ is the union of finitely many $S^{(i)}$, where we use the notation from 1.5, *c*. Therefore, if $\mathfrak{Z}'$ denotes the full subcategory of $\mathfrak{Z}$ consisting of all $\underline{Z} = (Z, Z_0, \eta_0) \in \mathfrak{Z}$ with $\eta_0(Z_0) \subset S_0$, $\mathfrak{Z}'$ will cover S_0. Moreover $\mathcal{G}$ induces an extension of $\mathcal{E}_0 = \mathcal{F}_0|_{S_0}$ to $\mathfrak{Z}'$, again denoted by $\mathcal{G}$. By induction we may assume that there exists a compactification S' of S_0 and an extension $\mathcal{E}$ of $\mathcal{E}_0$ to S', which is induced by $\mathcal{G}$ restricted to $\mathfrak{Z}'$. Blowing up W with centers in $S - S_0$, we may assume that S' is the closure of S_0 in W.

Let $\mathscr{B}$ be any coherent extension of $\mathscr{F}_0$ to W. Blowing up Fitting ideals we may assume that $\mathscr{B}$ is locally free. Since moreover, after blowing up, $W - W_0$ will become the exact support of an effective Cartier divisor, we can choose $\mathscr{B}$ as large as we want. Hence we can obtain:

b) There exists a locally free sheaf $\mathscr{B}$ on W with $\mathscr{B}|_{W_0} = \mathscr{F}_0$ such that $\mathscr{G} \subset \delta^*\mathscr{B}$ and $\mathscr{E} \subset \mathscr{B}|_{S'}$, where, of course, all inclusions coincide with the given isomorphisms on W_0.

Both, *a* and *b* are compatible with further blowing ups and we may assume in addition:

c) $S = S'$ is the closure of S_0 in W and $\delta^{-1}(S) \cup \delta^{-1}(W - W_0)$ is a normal crossing divisor. If T denotes the closure of $\delta^{-1}(S_0)_{\mathrm{red}}$ in W', then

$$\underline{T} = (T, T_0 = W'_0 \cap T, \eta_0 = \delta_0|_{T_0}) \in \mathscr{Z}'.$$

If we write $\eta = \delta|_T$, then $\mathscr{G}_{\underline{T}} = \eta^*\mathscr{E}$. Consider the restriction maps $\varphi: \mathscr{B} \to \mathscr{B}|_S$ and $\varphi': \delta^*\mathscr{B} \to \delta^*\mathscr{B}|_T$. We define $\mathscr{B}' = \varphi^{-1}(\mathscr{E})$ and $\mathscr{K} = \varphi'^{-1}(\mathscr{G}_{\underline{T}})$. If we write again $\delta'(\mathscr{B}')$ instead of $\delta^*(\mathscr{B}')$/torsion, we have an inclusion $\delta'(\mathscr{B}') \to \mathscr{K}$. Since $\mathscr{G}|_T = \mathscr{G}_{\underline{T}}$, we have a second inclusion $\mathscr{G} \to \mathscr{K}$, and both are isomorphisms over W'_0. Replacing W' by some blowing up, we may assume that $\delta'(\mathscr{B}')$ is invertible and (by abuse of notations) $\mathscr{K}$ as well. Moreover, by construction

$$\delta'(\mathscr{B}')|_T = \mathscr{K}|_T = \mathscr{G}|_T = \eta^*\mathscr{E}.$$

Therefore we can find some effective divisor E on W', not meeting T and contained in $\delta^{-1}(S - S_0)$ such that

$$\mathscr{G} \to \mathscr{K} \to \delta'(\mathscr{B}') \otimes \mathscr{O}_{W'}(E).$$

Let us apply Lemma 1.9. Using the notations introduced there, we may assume that $\delta^{-1}\sigma^{-1}(W - W_0)$ is a normal crossing divisor on V' and that $\sigma'(\mathscr{B}')$ is locally free. Then V' comes from some $\underline{V}' \in \mathscr{Z}$ and

$$\mathscr{G}_{\underline{V}'} = \varepsilon^*\mathscr{G} \subset \varepsilon^*\delta'\mathscr{B}' \otimes \rho^*\mathscr{O}_V(D) = \rho^*(\tau'\mathscr{B}' \otimes \mathscr{O}_V(D)).$$

After renaming we may replace b by the assumption
b') $\mathscr{G}$ is contained in $\delta^*\mathscr{B}$, $\mathscr{E} = \mathscr{B}|_S$ and $\mathscr{G}|_T = \delta^*\mathscr{B}|_T$.

Now let E be the support of $\delta^*\mathscr{B}/\mathscr{G}$. Again E is a divisor not meeting T. By 1.9, again, we can blow up W and W' such that E will become the pullback of some Cartier divisor D on W which is not meeting S. In other words, we can assume that for some neighbourhood U of S, the inclusion $\mathscr{G} \to \delta^*\mathscr{B}$ is an isomorphism over $\delta^{-1}(U)$. Glueing together $\mathscr{G}|_{W' - \delta^{-1}(S)}$ and $\mathscr{B}|_U$ along $\delta^{-1}(U) - \delta^{-1}(S) \simeq U - S$, we obtain an extension $\mathscr{F}$ of $\mathscr{F}_0$ to W. By construction $\delta^*\mathscr{F} = \mathscr{G}$ and, since all $\mathscr{G}_{\underline{Z}}$ are compatible, $\eta^*\mathscr{F} = \mathscr{G}_{\underline{Z}}$ for all $\underline{Z} \in \mathscr{Z}_W$.

This ends the proof of 1.6. For 1.7 we still have to prove 1.8. Before doing so we need some method to enforce the condition *a* of 1.8:

Lemma 1.10. *Let $\tau: W'_0 \to Y_0$ be a dominant finite morphism of reduced schemes and W'_0 normal. Then there exists a birational morphism $\eta: W'_0 \to W_0$ and a finite*

morphism $\tau_0: W_0 \to Y_0$ *such that* $\tau = \tau_0 \circ \eta$ *and:*

a) *The trace map from* $\tau_{0*}\mathcal{O}_{W_0}$ *to the integral closure* $\mathcal{O}_{\tilde{Y}_0}$ *of* $\mathcal{O}_{Y_0}$ *factors over* $\mathcal{O}_{Y_0}$.
b) *If* Y_1 *is an open subscheme of* Y_0, $W_1' = \tau^{-1}(Y_1)$, $\tau_1: W_1 \to Y_1$ *a morphism satisfying a) and if* $\tau|_{W_1'}$ *factors like*

$$W_1' \to W_1 \xrightarrow{\tau_1} Y_1$$

then the restriction of τ_0 *to* $\tau_0^{-1}(Y_1)$ *factors like* $\tau_0^{-1}(Y_1) \to W_1 \xrightarrow{\tau_1} Y_1$.

Proof. Let us choose $\mathcal{N}$ to be the sheaf of all local sections s of $\tau_*\mathcal{O}_{W_0'}$ such that the traces of all local sections of $s \cdot \tau_*\mathcal{O}_{W_0'}$ lie in $\mathcal{O}_{Y_0}$. Then $\mathcal{N}$ contains $\mathcal{O}_{Y_0}$ and is obviously an $\mathcal{O}_{Y_0}$ algebra. $\mathcal{N}$ contains $J \cdot \tau_*\mathcal{O}_{W_0'}$ for any ideal sheaf J of $\mathcal{O}_{Y_0}$ with $J \cdot \mathcal{O}_{\tilde{Y}_0} \subset \mathcal{O}_{Y_0}$. Therefore the integral closure of $\mathcal{N}$ is $\tau_*\mathcal{O}_{W_0'}$. So we found on $\mathcal{N}$ such that $W_0 = \mathrm{Spec}_{Y_0}(\mathcal{N})$ satisfies a. We can replace $\mathcal{N}$ by any larger subalgebra of $\tau_*\mathcal{O}_{W_0'}$, as long as the trace of all local sections s of $\mathcal{N}$ lies in $\mathcal{O}_{Y_0}$. Then, if $\tau_1: W_1 \to Y_1$ is as in b, we can assume that $\tau_*\mathcal{O}_{W_1}$ is contained in $\mathcal{N}|_{Y_1}$.

Proof of 1.8. Let $\tau_0: W_0 \to Y_0$ be any finite morphism and $\mathcal{Z}'$ as in 1.8, b.

a) If $\mathcal{Z}$ covers Y_0 then $\mathcal{Z}'$ covers W_0. Moreover, if $\mathcal{Z}$ covers Y_0 birationally, then the same holds for $\mathcal{Z}'$.

In fact, we just have to choose the generators $\underline{Z}^{(i)}$ of $\mathcal{Z}$ (as defined in 1.5c) such that $\tau_0^{-1}(S^{(i)})$ is non singular.

If $j: U_0 \to Y_0$ is an open subscheme we will write $j^*\mathcal{Z}$ for the category of all compactifying triples $\underline{Z}' = (Z', Z_0', \eta_0')$ of U_0 for which there exists some $\underline{Z} \in \mathcal{Z}$ with $Z = Z'$, $Z_0' = \eta_0^{-1}(U_0)$ and $\eta_0' = \eta_0|_{Z_0'}$. Choosing the generators $\underline{Z}^{(i)}$ of $\mathcal{Z}$ such that $U_0 = \bigcup_{i=\tau'}^{k} S^{(i)}$ for some τ' we obtain:

b) If $\mathcal{Z}$ covers Y_0 then $j^*\mathcal{Z}$ covers U_0.

Now let $\underline{Z}^{(i)}$ be generators of $\mathcal{Z}$ and let S_0 be the union of the closures of all $S^{(i)}$ for which $\eta_0^{(i)}$ is not birational. We may assume that all $Z^{(i)}$ are varieties, that

$$\dim(Z^{(1)}) = \dim(Z^{(2)}) = \ldots = \dim(Z^{(\tau)})$$

and that S_0 is the closure of $\bigcup_{i=1}^{\tau} S^{(i)}$. Moreover we may assume that $\mathbb{C}(Z^{(i)})$ is a Galois extension of $\mathbb{C}(S^{(i)})$. We can find an open affine subscheme Y_1 of Y_0 containing all general points of components of Y_0 and of the $S^{(i)}$ for $i = 1, \ldots, \tau$.

We can choose primitive polynomials $G_i(t)$ for $\mathbb{C}(Z^{(i)})$ whose coefficients are regular functions on $S^{(i)} \cap Y_1$. If $Y_1 \subset \mathbb{A}^m$ is an embedding, those $G_i(t)$ lift to some $F_i(t) \in \mathbb{C}[x_1 \ldots x_m, t]$. Let $\gamma: B \to \mathbb{A}^m$ be a finite Galois cover such that all $F_i(t)$ for $i = 1, \ldots, \tau$ split over $\mathbb{C}(B)$ in linear factors.

Define τ_1 to be the restriction of γ to $W_1 = \gamma^{-1}(Y_1)$. Then, by construction, the trace map from $\tau_{1*}\mathcal{O}_{W_1}$ maps to $\mathcal{O}_{Y_1}$. By Lemma 1.10 we choose $\tau_0: W_0 \to Y_0$, where W_0' is the union of the normalizations of the components of Y_0 in the function fields of the components of W_1. Let S' be the closure of an irreducible component of $\tau_0^{-1}(S^{(i)})$ in W_0 for $i \in \{1, \ldots, \tau\}$. By construction $G_i(t)$ will have a root in $\mathbb{C}(S')$ and

$\mathbb{C}(S')$ will contain $\mathbb{C}(Z^{(i)})$. Therefore we can find some $\underline{Z} \in \mathscr{Z}$, such that $\eta_0 : Z_0 \to Y_0$ factors over a birational map $\eta_0' : Z_0 \to S'$. We can add those to a generator system $\underline{Z}^{(i)'}$ of $\mathscr{Z}'$. As we have seen $j'^*\mathscr{Z}'$ covers $\tau_0^{-1}(U_0)$ birationally and since we may choose a system of generators of $\mathscr{Z}'$ containing those of $j'^*\mathscr{Z}'$ we have for $\mathscr{Z}'$ a generator system $\underline{Z}^{(i)'}$ such that $\dim(Z^{(i)'}) < \dim(Y_0 - U_0)$, whenever $\eta_0^{(i)'}$ is not birational.

§ 2. Weak positivity, again

2.1. Let Y_0 be a reduced quasi-projective scheme, $\mathscr{H}_0$ an ample invertible sheaf on Y_0, U an open subscheme of Y_0 and $\mathscr{F}_0$ a coherent torsion free sheaf on Y_0.

If $i : V \to Y_0$ is the largest open subscheme such that $i^*\mathscr{F}_0$ is locally free, we define $S^\mu(\mathscr{F}_0) = i_* S^\mu(i^*\mathscr{F}_0)$ and $\Lambda^\mu(\mathscr{F}_0) = i_* \Lambda^\mu(i^*\mathscr{F}_0)$. Recall

Definition. a) $\mathscr{F}_0$ is called *globally generated by A over U* if A is a subspace of $H^0(Y_0, \mathscr{F}_0)$ and $A \otimes_{\mathbb{C}} \mathscr{O}_U \to \mathscr{F}_0|_U$ surjective.

b) $\mathscr{F}_0$ is called *globally generated over U* if $\mathscr{F}_0$ is globally generated by $H^0(Y_0, \mathscr{F}_0)$ over U.

c) $\mathscr{F}_0$ is called *weakly positive over U* if $\mathscr{F}_0|_U$ is locally free and if for all $a > 0$ we can find some $b > 0$ such that $S^{a \cdot b}(\mathscr{F}_0) \otimes \mathscr{H}_0^b$ is globally generated over U.

The definition of weakly positive is independent of the ample sheaf $\mathscr{H}_0$ choosen. If $\mathscr{F}_0$ is locally free on Y_0 and Y_0 projective, then "$\mathscr{F}_0$ weakly positive over Y_0" is the same as "$\mathscr{F}_0$ semi-positive" in the sense of Kawamata [5].

For technical reasons we want to "bound the poles" of the sections needed in 2.1, c in order to generate the sheaf over U. May be, one can get along without, but in any case it seems to be of interest to keep control, where those sections are coming from. Also it seems that this boundedness makes it easier to glue local informations together (see 2.11). N. Nakayama gave an example of an invertible sheaf, where the boundedness does not follow from the weak positivity. Hence, we add it to the definition:

Definition 2.2. Keeping the notations from 2.1, we will call $\mathscr{F}_0$ *weakly positive over U with respect to* $(Y', \mathscr{F}')$ if the following conditions hold:

i) Y' is a projective non singular scheme, $\mathscr{F}'$ a coherent torsion free sheaf on Y' and $\mathscr{F}_0|_U$ is locally free.

ii) Y' is obtained as a desingularization $\delta : Y' \to Y$ of a projective compactification $j : Y_0 \hookrightarrow Y$.

iii) For all $\eta > 0$, one has an inclusion

$$\varphi_\eta : S^\eta(\mathscr{F}_0) \to j^* \delta_* S^\eta(\mathscr{F}') .$$

iv) Given an ample invertible sheaf $\mathscr{H}$ on Y and $a > 0$ we can find some $b > 0$ such that $S^{a \cdot b}(\mathscr{F}_0) \otimes j^*\mathscr{H}^b$ is globally generated over U by

$$A = H^0(Y, \delta_* S^{a \cdot b}(\mathscr{F}') \otimes \mathscr{H}^b) \cap H^0(Y_0, S^{a \cdot b}(\mathscr{F}_0) \otimes j^*\mathscr{H}^b)$$

(Of course, the intersection takes place in $H^0(Y_0, j^*\delta_* S^{a\cdot b}(\mathscr{F}') \otimes j^*\mathscr{H}^b)$ using $\varphi_{a\cdot b}$).

2.3. Some properties. If $j: Y_0 \to Y$ is a compactification and $n: \tilde{Y} \to Y$ the normalization, we write $\mathscr{A}_Y = j_*\mathscr{O}_{Y_0} \cap n_*\mathscr{O}_{\tilde{Y}}$. $\mathscr{A}_Y$ is an $\mathscr{O}_Y$-algebra, quasi-coherent, as the intersection of quasi-coherent sheaves, and coherent, as subsheaf of the coherent sheaf $n_*\mathscr{O}_{\tilde{Y}}$.

Let us write $(\ )^{**}$ instead of $S^1(\ )$ for the reflexive hull on Y'. Let $Y' \xrightarrow{\delta} Y \xleftarrow{j} Y_0$ be as in 2.2 i, ii and let $\sigma: Y'' \to Y'$ be a birational morphism with Y'' non singular and projective.

a) We can, in 2.2, assume that $\mathscr{A}_Y = \mathscr{O}_Y$.

b) If $\mathscr{A}_Y = \mathscr{O}_Y$ and if $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$, then $\mathscr{F} = j_*\mathscr{F}_0 \cap \delta_*(\mathscr{F}'^{**})$ is a coherent and torsion free sheaf, weakly positive over U.

c) If $\mathscr{F}_0$ is weakly positive over U with respect to $(Y'', \mathscr{F}'')$, then the same holds with respect to $(Y', \sigma_*\mathscr{F}'')$.

d) If $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$, then we can find some $\mathscr{F}''$ with $\sigma_* S^\eta(\mathscr{F}'') = S^\eta(\mathscr{F}')$. Especially $\mathscr{F}_0$ will be weakly positive over U with respect to $(Y'', \mathscr{F}'')$.

Proof. For a we just have to replace Y by $\mathrm{Spec}_Y(\mathscr{A}_Y)$.

In b it is obvious that $\mathscr{F}$ is coherent and torsion free. For simplicity we can write $\mathscr{F}'$ instead of $\mathscr{F}'^{**}$. For A as in 2.2., iv, we have to show that $A \otimes \mathscr{O}_Y$ maps to $S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b$. By abuse of notations we may assume that $\mathscr{F}$ is locally free. For $\eta > 0$ we have, for $\delta': Y' \to \tilde{Y}$ with $\delta = n \circ \delta'$,

$$S^\eta(n^*\mathscr{F}) = S^\eta(n^* j_*\mathscr{F}_0 \cap \delta'_*\mathscr{F}') = n^* S^\eta(j_*\mathscr{F}_0) \cap S^\eta(\delta'_*\mathscr{F}') .$$

Outside of the center of δ' both, $\delta'_* S^\eta(\mathscr{F}')$ and $S^\eta(\delta'_*\mathscr{F}')$ coincide and we get a map from the first sheaf to the second. Then $A \otimes \mathscr{O}_Y$ maps to

$$n_* n^* S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b = S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b \otimes n_*\mathscr{O}_Y$$

and to

$$j_* S^{a\cdot b}(\mathscr{F}_0) \otimes \mathscr{H}^b = S^{a\cdot b}(\mathscr{F}) \otimes \mathscr{H}^b \otimes j_*\mathscr{O}_{Y_0}.$$

Since $\mathscr{A}_Y = \mathscr{O}_Y$ we are done.

c) follows since we have again a natural map $\sigma_* S^\eta(\mathscr{F}'') \to S^\eta(\sigma_*\mathscr{F}'')$.

For d we choose some effective divisor E in the exceptional locus of σ such that $\mathscr{O}_{Y''}(-E)$ is relatively ample for σ. For some $\mu \gg 0$ and $\mathscr{F}'' = \sigma^*\mathscr{F}'$/torsion the sheaf $\mathscr{F}''(\mu \cdot E)|_E$ will be contained in $\oplus\, \mathscr{O}_E$. Then $S^\eta(\mathscr{F}''(\mu \cdot E))$ will have the same property. Therefore $\sigma_*(S^\eta(\mathscr{F}''(\mu \cdot E)) \otimes \mathscr{O}(\nu \cdot E))$ is for all $\nu > 0$ the same as $\sigma_*(S^\eta(\mathscr{F}''(\mu \cdot E)))$. Replacing $\mathscr{F}''$ by $\mathscr{F}''(\mu \cdot E)$ we found $\mathscr{F}''$.

In §3 of [14] and §5 of [9] the reader can find several properties of weakly positive sheaves. They carry over, in an obvious way, to "weakly positive with

respect to". In any case, the only non trivial properties we will need are:

2.4. Let us fix some notations:

$$\begin{array}{ccccc} Z_0 & \overset{i}{\rightarrow} & Z & \overset{\sigma}{\leftarrow} & Z' \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ Y_0 & \underset{j}{\rightarrow} & Y & \underset{\delta}{\leftarrow} & Y' \end{array} \qquad (*)$$

is supposed to be a commutative diagram with: i and j are projective compactifications, σ and δ desingularizations and τ_0 is a projective morphism of reduced schemes.

Functorial properties. Assume that $\mathscr{F}_0$ is locally free in some neighbourhood of the non normal locus of Y_0.

a) Assume that the morphism τ_0 in $(*)$ satisfies: $V = \tau_0^{-1}(U) \to U$ is finite, dominant and the trace map from $\tau_{0*}\mathscr{O}_V$ to the integral closure of $\mathscr{O}_U$ factors over $\mathscr{O}_U$. Then $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$, if $\tau_0^*\mathscr{F}_0$ is weakly positive over V with respect to $(Z', \tau'^*\mathscr{F}')$.

b) $\mathscr{F}_0$ is weakly positive over U with respect to $(Y', \mathscr{F}')$ if for some $\mu > 0$ one has: For all diagrams $(*)$ with τ_0 finite and for all ample invertible sheaves $\mathscr{A}$ on Z, the sheaf $\tau_0^*\mathscr{F}_0 \otimes i^*\mathscr{A}^\mu$ is weakly positive over $\tau_0^{-1}(U)$ with respect to $(Z', \tau^*\mathscr{F}' \otimes \sigma^*\mathscr{A}^\mu)$.

c) Assume that Y is projective and $\mathscr{F}$ locally free on Y, and let $\delta: Y' \to Y$ be a desingularization. Then $\delta^*\mathscr{F}$ is weakly positive over Y', if and only if $\mathscr{F}$ is weakly positive over Y with respect to $(Y', \delta^*\mathscr{F})$.

Proof. c) The "with respect to $(Y', \mathscr{F}')$", is automatically satisfied if $\mathscr{F}$ is known to be weakly positive over Y. Hence c is a special case of [14], 3.4 *e*. However, a simple proof is as follows: by [14], 3.4. *d*, one may assume $\mathscr{F}$ to be an invertible sheaf. Seshadri's numerical criterion for ampleness together with [14], 3.2, show that it is enough to use the obvious fact, that $\mathscr{F}$ is numerically effective if and only if the same holds for $\delta^*\mathscr{F}$.

a) and b) are nearly the same as [14], 3.4, b and c. The proof uses the same arguments. One just has to add, that the trace maps on Y and Y' are compatible. For example, assume that $\tau_0^*\mathscr{F}_0$ is weakly positive over V with respect to $(Z', \tau'^*\mathscr{F}')$. We may assume by 2.3, a that $\mathscr{A}_Y = \mathscr{O}_Y$ and moreover that $\mathscr{F}_0$ is locally free. As in 2.3, the sheaf $\sigma_*\tau'^*\mathscr{F}' \cap i_*\tau_0^*\mathscr{F}_0$ is weakly positive over V.

If one chooses $\mathscr{H}$ and E and $b \gg 0$ as in [14], 3.4, b one finds a map, surjective over U,

$$\oplus\, \tau_*\mathscr{O}_Z \otimes \mathscr{H}^b \to S^{2\cdot a\cdot b}(\sigma_*\tau'^*\mathscr{F}' \cap i_*\tau_0^*\mathscr{F}_0) \otimes \mathscr{H}^{2b} \otimes \tau_*\mathscr{O}_Z(-b\cdot E)\,.$$

The trace maps induce a map, surjective over U, from the right hand side to $S^{2\cdot a\cdot b}(j_*\mathscr{F}_0) \otimes \mathscr{H}^{2\cdot b}$ and another map, compatible with the first one, to

$$S^{2\cdot a\cdot b}(\delta_*\mathscr{F}') \otimes \mathscr{H}^{2\cdot b}\,.$$

The same argument proves b.

Later, the reference sheaves $\mathscr{F}'$ on Y' will come from direct images of powers of dualizing sheaves. The following "base change lemma", similar to [12], 3.2, will allow to compare different compactifications and desingularizations.

Lemma 2.5. *Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{\delta} & Z = Y'' \times_{Y'} X' & \xrightarrow{p_2} & X' \\ f'' & \searrow & \downarrow p_1 & & \downarrow f' \\ & & Y'' & \xrightarrow[\rho]{} & Y' \end{array}$$

be a commutative diagram of proper morphisms between reduced schemes, where δ is birational X'', Y'', X' and Y' are non singular, and $\rho' = p_2 \circ \delta$. Let $\mathscr{L}'$ be an invertible sheaf on X' and $\mathscr{L}'' = \rho'^* \mathscr{L}'$. Then for $\nu \geqq 1$ we have:

a) *If ρ is birational, then*

$$f'_*(\omega^{\nu}_{X'/Y'} \otimes \mathscr{L}') \quad \textit{contains} \quad \rho_* f''_*(\omega^{\nu}_{X''/Y''} \otimes \mathscr{L}'') .$$

b) *If ρ is finite then*

$$\rho^*(f'_*(\omega^{\nu}_{X'/Y'} \otimes \mathscr{L}'))^{**} \quad \textit{contains} \quad (f''_*(\omega^{\nu}_{X''/Y''} \otimes \mathscr{L}''))^{**} .$$

Proof. a) is trivial since one has an inclusion $\mathscr{O}_{Y''} \to \omega^{\nu}_{Y''/Y'}$ and

$$\rho'_*(\omega^{\nu}_{X''/Y'} \otimes \mathscr{L}'') = \omega^{\nu}_{X'/Y'} \otimes \mathscr{L}' .$$

b) Let Z' be the normalization of Z and let $X'' \xrightarrow{\delta''} Z' \xrightarrow{\delta'} Z$ be the factorization of δ. Since ρ is flat, the pullback of a reflexive sheaf under ρ remains reflexive and, leaving out codimension two subschemes of Y', we may assume that f' is flat and Gorenstein. Then Z will be Gorenstein and $\omega_{Z/Y''} = p_2^* \omega_{X'/Y'}$. If $\omega_{Z'}$ denotes the reflexive hull of the canonical sheaf on the smooth locus, one has

$$\delta'_* \omega_{Z'} = \mathscr{H}\mathrm{om}(\delta'_* \mathscr{O}_{Z'}, \omega_Z) \to \omega_Z$$

and a surjection $\delta'^* \delta'_* \omega_{Z'} \to \omega_{Z'}$. Since ω_Z is invertible we obtain $\omega_{Z'} \to \delta'^* \omega_Z$ and, for some exceptional divisor E of δ'' an inclusion $\omega_{X''}^{\nu-1} \to \delta^* \omega_Z^{\nu-1}(E)$. Altogether

$$\begin{aligned} \delta_*(\omega^{\nu}_{X''} \otimes \mathscr{L}'') &\to \delta'_*(\delta''_* \omega_{X''}(E) \otimes \delta'^*(\omega_Z^{\nu-1} \otimes p_2^* \mathscr{L}')) \to \\ &\to \delta'_* \omega_{Z'} \otimes \omega_Z^{\nu-1} \otimes p_2^* \mathscr{L}' \to \omega_Z^{\nu} \otimes p_2^* \mathscr{L}' . \end{aligned}$$

By flat base change we get

$$f''_*(\omega^{\nu}_{X''/Y''} \otimes \mathscr{L}'') \to p_{1*} p_2^*(\omega^{\nu}_{X'/Y'} \otimes \mathscr{L}) = \rho^*(f'_*(\omega^{\nu}_{X'/Y'} \otimes \mathscr{L})) .$$

Assumptions 2.6. Let

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{\sigma} & X' \\ \downarrow f_0 & & \downarrow f & & \downarrow f' \\ Y_0 & \xrightarrow{j} & Y & \xleftarrow{\delta} & Y' \end{array}$$

be a commutative diagram of morphisms of reduced quasi-projective schemes such that:

a) i and j are compactifications and σ and δ desingularizations.
b) f and f' are surjective and $X_0 = f^{-1}(Y_0)$.
c) f_0 is flat and all fibers of f_0 are reduced normal varieties of dimension n with at most rational Gorenstein singularities.

Theorem 2.7. *We keep the notations and assumptions from* 2.6. *Let* $\nu \geqq 1$ *be a natural number. Assume that one of the following conditions hold:*

i) $\nu = 1$.
ii) f_0 *is smooth and for some* $N > 0$ *the map* $f_0^* f_{0*} \omega^N_{X_0/Y_0} \to \omega^N_{X_0/Y_0}$ *is surjective.*
iii) $f_{0*}\omega^{\nu \cdot \mu}_{X_0/Y_0}$ *is locally free and commutes with arbitrary change for* $\mu > 0$. *Moreover* $f_0^* f_{0*}\omega^N_{X_0/Y_0} \to \omega^N_{X_0/Y_0}$ *is surjective for some* $N > 0$. *Then* $f_{0*}\omega^{\nu}_{X_0/Y_0}$ *is weakly positive over* Y_0 *with respect to* $(Y', f'_*\omega^{\nu}_{X'/Y'})$.

Remarks 2.8. 1. Since the total space of a deformation of rational Gorenstein singularities over a smooth base has again rational Gorenstein singularities, the assumption 2.6 implies:
c') The general fiber of f_0 is normal, f_0 is Gorenstein and, if $Y'_0 \to Y_0$ is a desingularization and $W'_0 \to Y'_0$ a non singular finite cover, then $X_0 \times_{Y_0} W'_0$ is normal with at most rational Gorenstein singularities. Moreover, the singular locus of $X_0 \times_{Y_0} W'_0$ will not contain a whole fiber of pr_2.

2. It follows from c' that for $Y'_0 = \delta^{-1}(Y_0)$ one has

$$f'_* \omega^{\nu}_{X'/Y'}|_{Y'_0} = \mathrm{pr}_2^* \omega^{\nu}_{X_0 \times_{Y_0} Y'_0 / Y'_0} = \delta^* f_{0*} \omega^{\nu}_{X_0/Y_0} .$$

Especially the condition iii of 2.2 holds.

3. It should be possible to replace in theorem 2.7 the condition c of 2.6 by the weaker condition c'.

4. The sheaf $f_{0*}\omega_{X_0/Y_0}$ is locally free and compatible with arbitrary base change. This result, due to Deligne [2], 5.5 in the smooth case, follows from an argument, which J. Kollár told me: By "Cohomology and Base Change" we only have to show that the sheaves $R^i f_{0*}\omega_{X_0/Y_0}$ are all locally free for all $i \geqq 0$. As well known one can find a complex $E^{\cdot}$ of locally free sheaves on Y_0 such that for all closed subschemes C of Y_0 and $W = X_0 \times_{Y_0} C$ the sheaves $R^i f_{0*}\omega_{X_0/Y_0} \otimes \mathcal{O}_W$ are given by the cohomology of $E^{\cdot} \otimes \mathcal{O}_C$. Therefore we may assume Y_0 to be a curve and, using flat base change, we can assume as well that Y_0 is smooth. Without changing $R^i f_{0*}\omega_{X_0/Y_0}$ we can replace X_0 by its desingularization and the local freeness follows from [6].

5. Using 4 (in fact only in the smooth case) we will show in 4.3 that ii) implies iii). Therefore, in the proof of 2.7 we will use base change whenever it is convenient.

Corollary 2.9. *Under the assumptions made in* 2.6 *assume that, for all fibers* F *of* f_0, ω_F *is semi-ample and of maximal Kodaira dimension. Then* $f_{0*}\omega^{\nu}_{X_0/Y_0}$ *is weakly positive over* Y_0 *for* $\nu \geqq 1$.

Proof. If ω_F is semi-ample then one obtains from the Grauert-Riemenschneider vanishing theorem (see also [3], §2) that $H^i(F, \omega_F^\nu) = 0$ for $i > 0$ and $\nu > 1$. Then $f_{0*}\omega^\nu_{X_0/Y_0}$ is compatible with base change. Since f_0 is flat, the Euler-Poincaré characteristic $\chi(\omega_F^\nu)$ is locally constant. As in 2.8,4 this follows since $H^\cdot(F, \omega_F^\nu)$ is given by a complex of locally free sheaves on Y_0, restricted to $f_0(F)$. Therefore $f_{0*}\omega^\nu_{X_0/Y_0}$ is locally free and compatible with base change for all $\nu \geqq 1$. Since ω_F is semi-ample for all fibres the assumption iii) of 2.7 holds true.

As an intermediate step we will obtain in §4 as well.

Theorem 2.10. *Under the assumptions made in* 2.6 *let* $\mathscr{L}'$ *and* $\mathscr{L}_0$ *be invertible sheaves on* X' *and* X_0 *respectively. Assume that* $\mathscr{L}_0 \subset i^*\sigma_*\mathscr{L}'$ *and that for some* $N > 0$ *the sheaf* $\mathscr{L}_0^N$ *is globally generated over* X_0 *by* $H^0(X', \mathscr{L}'^N) \cap H^0(X_0, \mathscr{L}_0^N)$. *Assume moreover that* $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ *is locally free and compatible with base change. Then* $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ *is weakly positive over* Y_0 *with respect to* $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

Lemma 2.11. a) *In order to prove* 2.7 *or* 2.10 *we may blow up* Y' *and* X'.
b) *In order to prove* 2.10 *it is enough to show that each point* $y \in Y_0$ *has an open neighbourhood* U *such that for the restriction* h *of* f_0 *to* $V = f_0^{-1}(U)$ *one has:* $h_*(\omega_{V/U} \otimes \mathscr{L}_0|_V)$ *is weakly positive over* U *with respect to* $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

Proof. a) Follows directly from 2.5, a.

b) Obviously it is enough to show that $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is weakly positive over U with respect to $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$. To this aim let us write $Y_0' = \delta^{-1}(Y_0)$, $X_0' = \sigma^{-1}(X_0)$ and

$$\begin{array}{ccccc} V & \xrightarrow{\beta} & X_0 & \xleftarrow{\sigma_0} & X_0' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_0'} \\ U & \xrightarrow[\alpha]{} & Y_0 & \xleftarrow[\delta_0]{} & Y_0' \end{array}$$

for the induced morphisms. Consider as in 2.3 $\mathscr{A}_Y = j_*\alpha_*\mathscr{O}_U \cap \delta_*\mathscr{O}_{Y'}$.

$\mathscr{A}_Y$ is a coherent $\mathscr{O}_Y$-algebra, isomorphic to $\mathscr{O}_{Y_0}$ over U. By 2.4,a and since $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is locally free and commuting with base change we are allowed to blow up Y as long as the center stays in $Y - U$. Therefore, replacing Y by $\mathrm{Spec}_Y(\mathscr{A}_Y)$ we can assume that $\mathscr{A}_Y = \mathscr{O}_Y$.

We may also assume that $\mathscr{L}'|_{X_0'} = \sigma_0^*\mathscr{L}_0$. In fact, if not we replace $\mathscr{L}'$ by the largest subsheaf with this property and blow X' up to make it invertible.

Since we assume that 2.10 holds for h we know from 2.3, b that

$$\mathscr{F} = \delta_* f'_*(\omega_{X'/Y'} \otimes \mathscr{L}') \cap (\alpha \circ j)_* h_*(\omega_{V/U} \otimes \mathscr{L}_0|_V)$$

is weakly positive over U. Moreover, $\mathscr{F}$ coincides with $h_*(\omega_{V/U} \otimes \mathscr{L}_0|_V)$ on U. Therefore it is enough to show that $j^*\mathscr{F}$ is contained in $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$. Obviously $j^*\mathscr{F}$ lies in $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0) \otimes \alpha_*\mathscr{O}_U$.

On the other hand, we have a natural map

$$f'_{0*}(\omega_{X_0'/Y_0'} \otimes \sigma_0^*\mathscr{L}_0) \to \mathrm{pr}_{2*}(\omega_{X_0 \times_{Y_0} Y_0'/Y_0'} \otimes \mathrm{pr}_1^*\mathscr{L}_0)$$

and, by assumption, the right hand side is $\delta_0^* f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$. Therefore we have

$$j^*\mathscr{F} \to \delta_{0*} f'_*(\omega_{X'_0/Y'_0} \otimes \sigma_0^* \mathscr{L}_0) \to f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0) \otimes \delta_{0*}\mathscr{O}_{Y'_0}$$

and, since $\mathscr{A}_Y = \mathscr{O}_Y$, we are done.

Remarks 2.12. a) The argument used in 2.11, b shows, how to use "with respect to $(Y', f'_*\omega^{\nu}_{X'/Y'})$" in 2.7 to extend "weak positivity" to sheaves coming from compactifications. For example, one gets easily, that under the assumptions made in 2.6, one also has:

Let Y_1 be an open subscheme of Y, containing Y_0 and f_1 the restriction of f to $X_1 = f^{-1}(Y_1)$. Assume that one of the following conditions holds true:
i) *f_1 is flat, Cohen Macauley and $f_{1*}\omega_{X_1/Y_1}$ locally free and compatible with base change. We have $\nu = 1$ and $\sigma_* \omega_{X'/Y'}|_{X_1}$ is larger than ω_{X_1/Y_1}.*
ii) *f_1 is flat, Gorenstein and $f_{1*}\omega^{\nu\cdot\mu}_{X_1/Y_1}$ is locally free and compatible with base change, for $\mu > 0$. Moreover, $\sigma_* \omega^{\nu\cdot\mu}_{X'/Y'}|_{X_1}$ is larger than $\omega^{\nu\cdot\mu}_{X_1/Y_1}$ and, for some $N > 0$, $f_1^* f_{1*}\omega^N_{X_1/Y_1} \to \omega^N_{X_1/Y_1}$ is surjective over X_0.*

Then $f_{1*}\omega^{\nu}_{X_1/Y_1}$ is weakly positive over Y_0 with respect to $(Y', f'_*\omega^{\nu}_{X'/Y'})$.

b) The assumption that, for $\nu > 1$, $f_0^* f_{0*}\omega^N_{X_0/Y_0}$ maps surjectively to $\omega^N_{X_0/Y_0}$ in 2.7, or the assumption "ω_F semi ample" in 2.9 can be replaced by some weaker condition in case that f_0 is smooth:

Assume that there exists an effective divisor D_0 on X_0, which has normal crossings relative to f_0, such that for some $N > 0$ the following two conditions are satisfied:

i) *For all $\mu > 0$ one has*

$$f_{0*}(\omega^{N\cdot\mu}_{X_0/Y_0} \otimes \mathscr{O}_{X_0}(-\mu \cdot D_0)) = f_{0*}\omega^{\mu\cdot N}_{X_0/Y_0}$$

ii)
$$f_0^* f_{0*}(\omega^N_{X_0/Y_0} \otimes \mathscr{O}_{X_0}(-D_0)) \to \omega^N_{X_0/Y_0} \otimes \mathscr{O}_{X_0}(-D_0)$$

is surjective (or: for all fibres F of f_0 $\omega^N_F \otimes \mathscr{O}_F(-D_0|_F)$ is semi ample and of maximal Iitaka dimension).

In other words, we ask for some very nice "Zariski-decomposition". The proof remains more or less the same. Just, in 2.10 and 3.16–24 one has to replace "$\mathscr{L}_0^N$ globally generated over X''_0 by $\mathscr{L}_0^N(-D_0)$ globally generated over X_0, and to work with $\mathscr{L}_0\left(-\left[\frac{D_0}{N}\right]\right)$ instead of $\mathscr{L}_0$. We leave it to the reader to check all details.

§ 3. Direct images of dualizing sheaves

We want to apply 1.6 and 1.7 to construct extensions of direct images of dualizing sheaves to certain compactifications. The starting point is:

Theorem 3.1 (W. Schmid, [11]). *Let $g'_0 : V'_0 \to W'_0$ be a proper surjective smooth morphism of non singular schemes such that all fibres are irreducible and of dimension n. Let $W'_0 \to W'$ be a non singular compactification such that $W' - W'_0$ is a normal crossing divisor. Assume that the monodromies of $R^n g'_{0*}\mathbb{C}_{V'_0}$ around the components*

of $W' - W'_0$ *are unipotent. Let* $\mathscr{H}'$ *be the canonical extension of* $\mathscr{H}'_0 = (R^n g'_{0*}\mathbb{C}_{V'_0}) \otimes_{\mathbb{C}} \mathscr{O}_{W'_0}$ *to* W' *(in the sense of Deligne, [2], II, 5.2). Then the subbundle* $\mathscr{F}'_0 = g'_{0*}\omega_{V'_0/W'_0}$ *of* $\mathscr{H}'_0$ *extends to a subbundle* $\mathscr{F}'$ *of* $\mathscr{H}'$.

Definition 3.2. For $g'_0: V'_0 \to W'_0$ as in 3.1, we will call $\mathscr{F}'$ the *Schmid extension* of $g'_{0*}\omega_{V'_0/W'_0}$ to W'. Hence, if we say that there *exists a Schmid extension* this means especially that the monodromy condition of 3.1 is satisfied.

Theorem 3.3 (Y. Kawamata, [5]). *Keeping the assumptions and notations from* 3.1, *let* $V'_0 \to V'$ *be a non singular compactification and* $g': V' \to W'$ *a morphism with* $g'|_{V'_0} = g'_0$. *Then*

a) $\mathscr{F}' = g'_*\omega_{V'/W'}$ *is the Schmid extension of* $g'_{0*}\omega_{V'_0/W'_0}$
b) $\mathscr{F}'$ *is weakly positive over* W'.

Kawamata's proof of b in [5] is quite complicated and uses the theory of variations of Hodge Structures. The Tankeev-Kollár vanishing theorem, applied to products of g'_0 with itself, gives a simpler proof of 3.3, b (see [6] or [13] for similar arguments).

3.4. If $f_0: X_0 \to Y_0$ is a smooth morphism of reduced schemes, all of whose fibers are compact manifolds of dimension n, we can construct a category of compactifying tripels $\mathscr{Z}$, which covers Y_0, and an extension $\mathscr{G}$ of $f_{0*}\omega_{X_0/Y_0}$ in the following way (see 1.5 for the notations):

3.5. Let S_0 be a closed subscheme of Y_0 and $\sigma_0: S'_0 \to S_0$ a desingularization. If $h'_0: T'_0 \to S'_0$ is the pullback of f_0 and $S'_0 \to S'$ a compactification such that $S' - S'_0$ is a normal crossing divisor, then the monodromy theorem ([2], III or [11]) says that the monodromy of $R^n h'_{0*}\mathbb{C}_{T'_0}$ around the components of $S' - S'_0$ are quasi-unipotent. Using for example [5], 18, or [9], 4.5, we find a non singular covering $\tau: W' \to S'$ such that $W'_0 = \tau^{-1}(S'_0)$ is the complement of a normal crossing divisor and the monodromies of $\tau^* R^n h'_{0*}\mathbb{C}_{T'_0}$ are unipotent.

Choose $\mathscr{Z}$ to be the smallest complete category of compactifying tripels which contains all the tripels $\underline{W}' = (W', W'_0, \sigma_0 \circ \tau|_{W'_0})$ obtained in this way.

3.6. In order to construct an extension $\mathscr{G}$ of $f_{0*}\omega_{X_0/Y_0}$ to $\mathscr{Z}$, it is enough to define $\mathscr{G}_{\underline{W}'}$ for the generators $\underline{W}'$ from 3.5. For those, if $g'_0: V'_0 \to W'_0$ is the pullback of f_0, $g'_{0*}\omega_{V'_0/W'_0}$ has a Schmid-extension $\mathscr{F}'$ and we define $\mathscr{G}_{\underline{W}'} = \mathscr{F}'$. In order to extend $\mathscr{G}$ to $\mathscr{Z}$ by pullback we need that the Schmid extensions are compatible.

Lemma 3.7. *Under the notations and assumptions of* 3.1, *let* $\eta: Z \to W'$ *be a proper morphism of non singular schemes and* $\eta^{-1}(W' - W'_0)$ *a normal crossing divisor. For* $Z_0 = \eta^{-1}(W'_0)$ *the morphism* $\mathrm{pr}_2: V'_0 \times_{W'_0} Z_0 \to Z_0$ satisfies again the assumptions of 3.1 and $\eta^*\mathscr{F}'$ *is the Schmid extension of* $\mathrm{pr}_{2*}\omega_{V'_0 \times_{W'_0} Z_0/Z_0}$.

Proof. This is just saying that the unipotency of the monodromy and the canonical extension $\mathscr{H}'$ of $\mathscr{H}'_0$ are compatible with pullbacks.

By 3.7 we can extend $\mathscr{G}$ to $\mathscr{Z}$ and found the extension asked for. By 1.7 we get:

Theorem 3.8. *Let $f_0\colon X_0 \to Y_0$ be a smooth morphism of reduced schemes, all of whose fibers are compact manifolds of dimension n. Then there exists a dominant finite covering $\tau_0\colon W_0 \to Y_0$, a compactification $i\colon W_0 \to W$ and a locally free sheaf $\mathcal{F}$ on W with:*

a) *The trace map from $\tau_{0*}\mathcal{O}_{W_0}$ to the integral closure $\mathcal{O}_{\tilde{Y}_0}$ of $\mathcal{O}_{Y_0}$ factors over $\mathcal{O}_{Y_0}$.*

b) $$\tau_0^* f_{0*}\omega_{X_0/Y_0} = i^*\mathcal{F}$$

c) *If $\delta\colon W' \to W$ is a desingularization and $W'_0 = \delta^{-1}(W_0)$ the complement of a normal crossing divisor, then $\delta^*\mathcal{F}$ is the Schmid extension, of $\mathrm{pr}_{2*}\omega_{X_0 \times_{Y_0} W'_0/W'_0}$.*

Corollary 3.9. *If f_0 is smooth and $v = 1$, then $f_{0*}\omega_{X_0/Y_0}$ is weakly positive over Y_0 and, moreover, theorem 2.7 holds true.*

In fact, 3.9 is a direct consequence of 3.8 and 2.4, a. To keep control of the reference sheaf on Y' one has to use 3.3, b and 2.5. Since the exact formulation is repeated in the proof of 3.24, we do not give the details.

In §4 we will show, that 2.11 and 2.7 can be deduced quite easily from 3.9 if f_0 is smooth. Hence, *the reader just interested in moduli of manifolds can skip the rest of this paragraph and go to* 4.2.

For the non smooth case we consider $f_{0*}(\omega_{V_0/W_0} \otimes \mathcal{L}_0)$ directly, allowing $\mathcal{L}_0$ to be trivial. To this aim we have to have a closer look to the Schmid extension first:

3.10. We want to show, that the Schmid extension of $g'_{0*}\omega_{V'_0/W'_0}$ is compatible with the Schmid extension of certain components of the boundary. To this aim, let g'_0 and $g'\colon V' \to W'$ be as in 3.1 and 3.3. We may assume that $\Sigma = g'^*(W' - W'_0)$ is a normal crossing divisor. Let Z be an irreducible component of $W' - W'_0$ and $E = (W' - W'_0) - Z$. Let Γ be a component of Σ of multiplicity one in Σ and dominant over Z. Write $W'_1 = W' - E$, $V_1 = g'^{-1}(W'_1)$, $\Gamma_0 = \Gamma \cap g'^{-1}(Z_0)$ for Z_0 open in Z, and assume that $Z - Z_0$ is a normal crossing divisor.

Write

$$\begin{array}{ccccccc} V'_1 & \to & V' & \leftarrow & \Gamma & \leftarrow & \Gamma_0 \\ \downarrow g'_1 & & \downarrow g' & & \downarrow \alpha & & \downarrow \alpha_0 \\ W'_1 & \to & W' & \leftarrow & Z & \leftarrow & Z_0 \end{array}$$

for the corresponding morphisms.

If $R = g'^*(Z) - \Gamma$, we have natural adjunction maps

$$\begin{array}{ccc} g'_*\omega_{V'/W'}(-R) & \overset{\Phi'}{\to} & \alpha_*\omega_{\Gamma/Z} \\ \downarrow & & \downarrow \eta \\ g'_*\omega_{V'/W'} & \underset{\Phi}{\to} & \alpha_*\omega_{\Gamma/Z}(R \cap \Gamma) \end{array}$$

Proposition 3.11. *Assume that $\alpha_0\colon \Gamma_0 \to Z_0$ is smooth and that the monodromy of $R^{n-1}\alpha_{0*}\mathbb{C}_{\Gamma_0}$ around the components of $Z - Z_0$ is unipotent, and that $\eta|_{Z_0}$ is an isomorphism. Then the image of Φ is $\alpha_*\omega_{\Gamma/Z}$.*

Remark 3.12. It is always true that Φ' is surjective,

Proof of 3.12. The question is local and we may assume therefore that W' is a projective manifold. For an ample invertible sheaf $\mathscr{H}$ on W', the Kollár-Tankeev vanishing theorem ([6], see also [3], §3) tells us that

$$H^0(V', \omega_{V'}(\Gamma) \otimes g'^* \mathscr{H}) \to H^0(\Gamma, \omega_\Gamma \otimes \alpha^* \mathscr{H}/_Z)$$

is surjective. Since this adjunction map is induced by Φ', and since we can choose $\mathscr{H}$ as ample as we want, Φ' must be surjective.

Proof of 3.11. Let us write $R = R_1 + R_2$, where R_1 is the sum of the components of R which are dominant over Z and R_2 the others.

In order to prove 3.11 we can blow up W' and replace it by finite coverings. In fact, since both, $g'^* \omega_{V'/W'}$ and $\alpha_* \omega_{\Gamma/Z}$ are Schmid extensions, they are compatible with pullbacks as well as the surjection, induced by Φ, from $g'_* \omega_{V'/W'}|_{W'-(Z-Z_0)}$ to $\alpha_{0*} \omega_{\Gamma_0/Z_0}$.

By 4.6 of [9] or 6.1 of [12] we can make semi-stable reduction in codimension one and, replacing Z_0 by some smaller open set, assume that R_1 is a reduced normal crossing divisor. Blowing up we may as well assume that $Z_0 = Z - (Z \cap E)$ and that g'_1 is flat. Making semi-stable reduction a second time we can finally assume that all components of Σ which are dominant over components of $W' - W'_0$ are reduced in Σ.

For some $b \gg 0$, Φ will induce a map

$$g'_* \omega_{V'/W'}(-b \cdot R_2) \to \alpha_* \omega_{\Gamma/Z}$$

and using Φ', we see that $b = 1$ is enough. Hence 3.11 follows from

Claim 3.13. Under the assumptions made above $g'_* \omega_{V'/W'}(-R_2) \to g'_* \omega_{V'/W'}$ is an isomorphism.

Proof. Write $R_2 = \sum_{\nu=1}^r \mu_\nu \cdot \Sigma_\nu$, for $\mu_\nu > 0$, and $\Sigma = \sum_{\nu=1}^k \eta_\nu \cdot \Sigma_\nu$. Since R_2 is supported in $g'^{-1}(E)$ we have $\mu_\nu < \eta_\nu$ for $\nu = 1, \ldots, r$. Let M be a number, larger than η_ν for $\nu = 1, \ldots, r$.

Since our question is local in W' we can write $\mathscr{O}_{W'}(E + Z) = \mathscr{H}^M$ for some invertible sheaf $\mathscr{H}$. Let S be the normalization of the covering, obtained by taking the M-th root out of $E + Z$, and T the normalization of that, obtained by taking the M-th root out of Σ, where for $\mathscr{L} = g'_* \mathscr{H}$ we write $\mathscr{L}^M = \mathscr{O}_{V'}(\Sigma)$ (see 3.14 and [9], §4 or [3], 2.7 for this well known construction).

Let

$$\begin{array}{ccc} T & \xrightarrow[\tau]{} & V' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g'} \\ S & \xrightarrow[\rho]{} & W' \end{array}$$

denote the induced morphisms. We have $\omega_S = \rho^* \omega_{W'} \otimes \mathscr{H}^{M-1}$ and

$$\tau_* \omega_T = \bigoplus_{i=0}^{M-1} \omega_{V'} \otimes \mathscr{L}^i \left(- \left[\frac{i \cdot \Sigma}{M} \right] \right).$$

Therefore $g'_*\tau_*\omega_{T/S}$ contains $g'_*\omega_{V'/W'} \otimes \mathcal{O}_{V'}\left(-\left[\frac{M-1}{M}\Sigma\right]\right)$ as a direct summand. This remains true if we replace S by some higher cover and T by the normalization of the pullback. Since S has at most quotient singularities we may, by abuse of notations, assume S to be non singular and $\delta^{-1}(E+Z)$ to be a normal crossing divisor. Since T has at most rational singularities $h_*\omega_{T/S}$ will again be a Schmid extension and hence locally free. Therefore

$$g'_*\left(\omega_{V'/W'} \otimes \mathcal{O}_{V'}\left(-\left[\frac{M-1}{M}\Sigma\right]\right)\right)$$

is locally free. This sheaf is contained in $g'_*\omega_{V'/W'}$ and, since all components of Σ, dominant over components of $W' - W'_0$, are of multiplicity one, both sheaves must be isomorphic. On the other hand,

$$\frac{M-1}{M}\cdot\eta_\nu \geqq \eta_\nu - 1 \geqq \mu_\nu.$$

Therefore $\left[\frac{M-1}{M}\Sigma\right] \geqq R_2$.

3.14. In the proof of 3.13 we used the coverings obtained by taking roots out of divisors. Recall:

Let Z_0 be a scheme, $\mathcal{L}$ an invertible sheaf, D an effective Cartier divisor and $N > 0$. If $\mathcal{L}^N = \mathcal{O}_{Z_0}(D)$ and if s is the corresponding section, then we call the flat finite morphism

$$\tau_0 \colon Z_1 = \mathrm{Spec}_{Z_0}\left(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/s^{-1}\right) \to Z_0$$

the *covering obtained by taking the N-th root out of D*

Properties 3.15. Assume that Z_0 is flat and Gorenstein over some scheme S_0.

a) Z_1 is a reduced scheme, flat and Gorenstein over S_0, and

$$\tau_{0*}\omega_{Z_1/S_0} = \bigoplus_{i=0}^{N-1} \mathcal{L}^i \otimes \omega_{Z_0/S_0}.$$

b) If $S_0 = \mathrm{Spec}\,\mathbb{C}$ and if Z_0 is normal with rational singularities, let us assume that $\mathcal{L}^N$ is generated by some finite dimensional vector space A of global sections and that $s \in A$ is a general section. Then Z_1 is again normal with at most rational Gorenstein singularities.

Proof. τ_0 is flat and Z_1 reduced. Moreover ω_{Z_1/S_0} is just the sheaf on Z_1 corresponding to the $\bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}$ module $\mathcal{H}om(\bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}, \omega_{Z_0/S_0})$. To prove b we consider a desingularization

$$\eta_0 \colon Z'_0 \to Z_0 \quad \text{and} \quad D' = \eta_0^* D\,.$$

since D was in general position D' will be non singular. Let Z'_1 be the covering

obtained by taking the N-th root out of D' and

$$\begin{array}{ccc} Z_1' & \xrightarrow[\tau_0']{} & Z_0' \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_0} \\ Z_1 & \xrightarrow[\tau_0]{} & Z_0 \end{array}$$

the induced morphisms. Then Z_1' is non singular and

$$\tau_{0*}\eta_{1*}\mathcal{O}_{Z_1'} = \eta_{0*}\tau_{0*}'\mathcal{O}_{Z_1'} = \eta_{0*}\bigoplus_{i=0}^{N-1}\eta_0^*\mathscr{L}^{-i} = \tau_{0*}\mathcal{O}_{Z_1}$$

gives the normality of Z_1. Z_1 can only have rational singularities since

$$\tau_{0*}\eta_{1*}\omega_{Z_1'/S_0} = \eta_{0*}\tau_{0*}'\omega_{Z_1'/S_0} = \eta_{0*}\bigoplus_{i=0}^{N-1}\omega_{Z_0'/S_0}\otimes\eta_0^*\mathscr{L}^{-i} = \tau_{0*}\omega_{Z_1/S_0}\,.$$

Assumptions 3.16. $f_0\colon X_0 \to Y_0$ is supposed to be a flat Gorenstein morphism of reduced quasi-projective schemes, $\mathscr{L}_0$ an invertible sheaf on X_0, D_0 an effective Cartier divisor and $N \geqq 1$. Assume that $\mathscr{L}^N = \mathcal{O}_{X_0}(D_0)$ where: If $N = 1$, then D_0 should be empty. If $D_0 = \varnothing$ then N should be as small as possible. Moreover, D_0 should not contain any fiber of f_0. Assume moreover:

i) $f_{0*}(\omega_{X_0/Y_0}\otimes\mathscr{L}_0)$ is locally free and commutes with arbitrary base change
ii) Let $\tau_0\colon X_1 \to X_0$ be the covering obtained by taking the N-th root out of D_0 and $f_1 = f_0\circ\tau_0\colon X_1 \to Y_0$. Then all fibers of f_1 are reduced normal varieties with at most rational (Gorenstein) singularities.

Remarks 3.17. a) If f_0 is smooth, we can replace ii) by:
ii') D_0 is a normal crossing divisor relative to f_0 and all components of D_0 have a multiplicity strictly smaller than N. Let $\tau_0\colon X_1 \to X_0$ be the normalization of the covering obtained by taking the N-th root out of D_0. Then all fibers of $f_1 = f_0\circ\tau_0$ should be reduced normal varieties.
b) As in 2.6 it should be enough to ask in ii) or ii') that f_1 satisfies the weaker assumption c' from 2.8, 1.

We have to define some kind of Schmid extension of $f_{0*}(\omega_{X_0/Y_0}\otimes\mathscr{L}_0)$. To this aim (and not just to increase confusion) let us formulate:

Statement 3.18. There exists a diagram of fiber products

$$\begin{array}{ccccc} V_1' & \to & V_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow{\scriptstyle \beta_0'} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \tau_0} \\ V_0' & \xrightarrow{\delta'} & V_0 & \xrightarrow{\pi'} & X_0 \\ \downarrow{\scriptstyle g_0'} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f_0} \\ W_0' & \xrightarrow{\delta_0} & W_0 & \xrightarrow{\pi_0} & Y_0 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} & & \\ W' & \xrightarrow{\delta} & W & & \end{array}$$

with $f_1 = f_0 \circ \tau_0$, $g_1 = g_0 \circ \beta_0$, $g'_1 = g'_0 \circ \beta'_0$, $\mathscr{M}_0 = \pi'^* \mathscr{L}_0$, $\mathscr{M}'_0 = \delta'^* \mathscr{M}_0$, $B_0 = \pi'^* D_0$ and $B'_0 = \delta'^* B_0$, such that:

a) π_0 is finite, dominant and the trace map from $\pi_{0*} \mathscr{O}_{W_0}$ to the integral closure of $\mathscr{O}_{Y_0}$ factors over $\mathscr{O}_{Y_0}$.
b) i and i' are projective compactifications and δ: $W' \to W$ is a desingularization such that $\delta^{-1}(W - W_0)$ is a normal crossing divisor.
c) There exists an open subscheme W'_s of W'_0 such that $W' - W'_s$ is a normal crossing divisor and a desingularization σ_s: $T'_s \to V'_s = g'^{-1}_1(W'_s)$ which is smooth over W'_s.

3.19. Assume that 3.16 holds true and that we have a diagram as in 3.1. Assume moreover that the smooth morphism $h_s = g'_1|_{V'_s} \circ \sigma_s$: $T'_s \to W'_s$ satisfies the assumptions made in 3.1 on the monodromies. Let $\mathscr{G}'$ be the Schmid extension of $h'_{s*} \omega_{T'_s/W'_s}$ to W'. By assumption the fibers of g'_1 have rational Gorenstein singularities. Therefore the same holds true for the total space V'_1. If σ: $T'_1 \to V'_1$ is any desingularization we will have by 3.3, a, and 3.15, a, that $\mathscr{F}'_0 = g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}'_0)$ is a direct summand of

$$g'_{1*} \omega_{V'_1/W'_0} = (g'_1 \circ \sigma)_* \omega_{T'_1/W'_0} = \mathscr{G}'|_{W'_0}.$$

Therefore we can define:

Definition 3.20. We say that $\mathscr{F}'_0 = g'_{0*}(\omega_{V'_0/W'_0} \otimes \mathscr{M}'_0)$ *has a Schmid extension*, if the assumptions of 3.18, b and c, and of 3.19 hold true and we call in that case $\mathscr{F}' = \mathscr{G}' \cap i'_* \mathscr{F}'_0$ *the Schmid extension* of $\mathscr{F}'_0$ (with respect to B'_0)

Properties 3.21.
i) $\mathscr{F}'$ is locally free and weakly positive over W'.
ii) $\mathscr{F}'$ is compatible with pullback under dominant morphisms.
iii) Let V' be a compactification of V'_0 and g': $V' \to W'$ an extension of g'_0 to V'. Let σ: $V'' \to V'$ be a desingularization, $g'' = g' \circ \sigma$ and σ_0 the restriction of σ to $V''_0 = \sigma^{-1}(V'_0)$. Assume that there exist an effective normal crossing divisor B'' and an invertible sheaf $\mathscr{M}''$ on V'' with

$$\mathscr{O}_{V''}(B'') = \mathscr{M}''^N, \; B''|_{V''_0} = \sigma_0^* B'_0 \quad \text{and} \quad \mathscr{M}''|_{V''_0} = \sigma_0^* \mathscr{M}'_0 .$$

Then

$$\mathscr{G}' = \bigoplus_{i=0}^{N-1} g''_* \left(\omega_{V''/W'} \otimes \mathscr{M}''^i \left(- \left[\frac{i \cdot B''}{N} \right] \right) \right)$$

and

$$\mathscr{F}' = g''_* \left(\omega_{V''/W'} \otimes \mathscr{M}'' \left(- \left[\frac{B''}{N} \right] \right) \right) \subseteq g''_* (\omega_{V''/W'} \otimes \mathscr{M}'') .$$

Proof. If T'_1 from 3.19 is choosen large enough, we can take a non singular compactification T' of T'_1 which is dominating V''. Then, for h': $T' \to W'$ the direct sum in iii) is nothing but $h'_* \omega_{T'/W'}$ (see for example [9], §4 or [3], 2.7), and $\mathscr{F}'$ must be the summand for $i = 1$. Since $\mathscr{F}'$ is a direct summand of $\mathscr{G}'$ we obtain i) from 3.3, b, and ii) from 3.7.

Corollary 3.22. *Let Z be a smooth irreducible divisor in W' such that $Z_0 = W'_0 \cap Z$ is the complement of a normal crossing divisor and let φ_0 be the restriction of g'_0 to*

$\Delta_0' = g_0'^{-1}(Z_0)$. *If* $\varphi_{0*}(\omega_{\Delta_0/Z_0} \otimes \mathscr{M}_0')$ *has a Schmid extension* $\mathscr{F}'_Z$ *to* Z, *with respect to* $B_0'|_{\Delta_0'}$, *then* $\mathscr{F}'|_Z = \mathscr{F}'_Z$.

Proof. Returning to the notations from 3.21, let us consider

$$\begin{array}{ccccccc}
\Gamma & \to & T' & & & & \\
\downarrow \beta & & \downarrow & & & & \\
\Delta & \to & V'' & \xrightarrow{\sigma} & V' & \leftarrow & \Delta' \\
\downarrow \rho & & \downarrow g'' & & \downarrow g' & & \downarrow \varphi \\
Z & \to & W' & = & W' & \leftarrow & Z
\end{array}$$

where Δ' is the closure of Δ_0 in V', Δ the proper transform of Δ' in V'', Γ the proper transform of Δ in T' and $\alpha = \rho \circ \beta$. We may assume that $\Delta + B''$ is a normal crossing divisor and that $\beta: \Gamma \to \Delta$ is a desingularization of the covering obtained by taking the N-th root out of $B''|_\Delta$. Then Γ will be a desingularization of a compactification of $\Delta_1' = g_1'^{-1}(Z_0)$. Write $\varphi_1': \Delta_1' \to Z_0$ for the induced map. $\mathscr{F}'_Z$ will be a direct summand of $\alpha_* \omega_{\Gamma/Z}$. We have a natural map

$$\phi_0: \mathscr{G}'|_{W_0'} = g_{1*}' \omega_{V_1'/W_0'} \to \varphi_{1*}' \omega_{\Delta_1'/Z_0} = \alpha_* \omega_{\Gamma/Z}|_{Z_0}$$

which induces an isomorphism $\mathscr{G}'|_{Z_0} \to \alpha_* \omega_{\Gamma/Z}|_{Z_0}$. If $Z_0 \cap W_s \neq \varnothing$ we are done. If not, this implies that the assumptions of 3.11 are satisfied and Φ_0 extends to a surjection $\mathscr{G}' \to \alpha_* \omega_{\Gamma/Z}$, or an isomorphism $\mathscr{G}'|_Z \to \alpha_* \omega_{\Gamma/Z}$.

Theorem 3.23. *Under the assumptions made in* 3.16 *we can construct a diagram satisfying* 3.18 *and a locally free sheaf* $\mathscr{F}$ *on* W *such that* $g_{0*}'(\omega_{V_0'/W_0'} \otimes \mathscr{M}_0')$ *has a Schmid extension* $\mathscr{F}'$ *with respect to* B_0' *and* $\mathscr{F}' = \delta^* \mathscr{F}$.

Proof. Of course we can find such a diagram, for example for $\pi_0 = \mathrm{id}$, such that all the properties asked for in 3.18 hold true. However, the morphism $h_s = g_1'|_{V_s'} \circ \sigma_s$ will not satisfy the assumption made in 3.1 on the monodromies. Using [5], 18, (see also [9], 4.5) we can find some non singular covering of W' with sufficiently high ramification along $W' - W_s'$ such that the monodromy condition holds true over this covering. Renaming and using 1.10 to get back W_0, we found a diagram 3.18, such that $g_{0*}'(\omega_{V_0'/W_0'} \otimes \mathscr{M}_0')$ has a Schmid extension.

We may repeat this construction for all closed subschemes of Y_0. Let $\mathscr{Z}$ be the smallest complete category of compactifying tripels, as defined in 1.5, which contains all those tripels $\underline{W}' = (W', W_0', \tau_0 \circ \delta_0)$ thereby obtained. Obviously $\mathscr{Z}$ covers Y_0.

By construction we have a Schmid extensions $\mathscr{F}'$ of $g_{0*}'(\omega_{V_0'/W_0'} \otimes \mathscr{M}_0')$ to W' (we keep the notation from 3.18, even if we start with a subscheme of Y_0). Let us denote it by $\mathscr{G}_{\underline{W}'}$. If $\tau: \underline{Z} = (Z, Z_0, \eta_0) \to \underline{W}'$ is a morphism in $\mathscr{Z}$ and $\tau: Z \to W'$ dominant, then $\tau^* \mathscr{G}_{\underline{W}'}$ is the Schmid extension on Z by 3.21. If $\tau(Z)$ is a smooth irreducible divisor in W', then the same holds true by 3.22, and, since we can blow up W' to desingularize $\tau(Z)$, a combination of both steps gives the same result, whenever $\tau(Z)$ is a divisor. By induction on the dimension of $\tau(Z)$ we find that $\tau^* \mathscr{G}_{\underline{W}'}$ always is the Schmid extension. Therefore, if we define for $\underline{Z} \in \mathscr{Z}$, $\mathscr{G}_{\underline{Z}}$ by

pullback, this is well defined and compatible with morphisms in $\mathscr{Z}$. Hence we found an extension $\mathscr{G}$ of $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ to $\mathscr{Z}$, as defined in 1.5, f.

By 1.7 we find a new covering W_0 of Y_0 satisfying 3.18, a, a compactification W of W_0 and a locally free sheaf $\mathscr{F}$ on W which is induced by $\mathscr{G}$. Finally, desingularizing W we will get back the whole diagram asked for in 3.23.

Corollary 3.24. *Under the assumptions made in* 3.16, *let* $i\colon X_0 \to X$ *and* $j\colon Y_0 \to Y$ *be compactifications,* $\delta\colon Y' \to Y$ *and* $\sigma\colon X' \to X$ *desingularizations and* $f'\colon X' \to Y'$ *a morphism induced by* f_0. *Let* $\mathscr{L}'$ *be an invertible sheaf of* X' *and* D' *an effective divisor with* $\mathscr{L}'^N = \mathscr{O}_{X'}(D')$. *Assume that* $\mathscr{L}'|_{\delta^{-1}(Y_0)} = \delta^*(\mathscr{L}_0)$ *and* $D'|_{\delta^{-1}(Y_0)} = \delta^*(D_0)$. *Then* $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ *is weakly positive over* Y_0 *with respect to* $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

Proof. We may use the diagram 3.18 existing by 3.23, and the compactification V' from 3.21, iii. Moreover we may assume that π_0 and π' induce morphisms $\tau\colon W' \to Y'$ and $\gamma\colon V' \to X'$. By 2.11, a, (or better, using the same simple argument) we are allowed to blow up Y' and hence we may assume that the ramification locus of τ in Y' is a normal crossing divisor. The normalization $\tilde{Y}$ of Y' in the function ring of W' need not be non singular, but by [5], 19, (see also [9], 4.7) there is a finite non singular covering dominating $\tilde{Y}$. Replacing W', V' etc, by the induced covering and using 1.10 to enforce condition 3.18 a, we may assume that W' is a desingularization of a non singular finite cover of Y'.

Since $\delta^*\mathscr{F} = \mathscr{F}'$ is weakly positive over W' we find by 2.4, c, that $\mathscr{F}$ is weakly positive over W with respect to $g''_*(\omega_{V''/W'} \otimes \mathscr{M}'')$, where, in the notations of 3.21, iii, $\mathscr{M}'' = \sigma^*\gamma^*\mathscr{L}'$. By 2.5, a and b, and 2.4, a we find $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ to be weakly positive over Y_0 with respect to $(Y', f'_*(\omega_{X'/Y'} \otimes \mathscr{L}'))$.

§4. Direct images of powers of dualizing sheaves

We want to prove theorem 2.7. The methods employed are similar to those used in [12] (see also [9], §5 and [14] §3).

4.1. *Proof of 2.10.* We will use the notations of 2.6 and 2.10.

Let $y \in Y_0$ be a given point. Since $\mathscr{L}_0^N$ is generated over X_0 by $A = H^0(X', \mathscr{L}'^N) \cap H^0(X_0, \mathscr{L}_0^N)$, we can take a general section of the image of A in $H_0(f_0^{-1}(y), \mathscr{L}_0^N|_{f_0^{-1}(y)})$ and lift it to $s \in A$. Let D_0 be the zero divisor of s on X_0 and X_1 the covering obtained by taking the N-th root out of D_0 (see 3.14). By 3.15, b, the fiber of $f_1\colon X_1 \to X_0 \xrightarrow{f_0} Y_0$ over y' will be a normal variety with rational Gorenstein singularities for all y' in a neighbourhood of y. By 2.11, b we are allowed to replace Y_0 by this neighbourhood. Hence we may assume that the assumptions made in 3.16 are all satisfied. By construction, the pullback of D_0 extends to some divisor D' on X' with $\mathscr{O}_{X'}(D') = \mathscr{L}'^N$. Then 3.24 implies the positivity, asked for in 2.10.

4.2. For those who did not want to read the constructions 3.10–3.24, we sketch a

slightly different proof of 2.10 in the smooth case:

Let $y \in Y_0$ be given and, again, $f_1 : X_1 \to Y_0$ the morphism obtained by taking the N-th root out of a general section

$$s \in H^0(X', \mathscr{L}'^N) \cap H^0(X_0, \mathscr{L}_0^N) .$$

If D_0 is the zero divisor of S, D_0 will be non singular and will meet all fibers over some neighbourhood of y transversally. By 2.11,b we can replace Y_0 by this neighbourhood.

Then however, $f_1 : X_1 \to Y_0$ is smooth. Moreover, $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ will be direct summand of $f_{1*}\omega_{X_1/Y_0}$. Hence 3.9 implies that $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is weakly positive. Since s extends to some section of $\mathscr{L}'^N$, we can extend the covering X_1 of X_0 to some compactification and obtain 2.10 as stated.

Corollary 4.3. *If $f_0 : X_0 \to Y_0$ is a smooth morphism and $\mathscr{L}_0$ a sheaf such that $f_0^* f_{0*} \mathscr{L}_0^N \to \mathscr{L}_0^N$ is surjective, for some $N > 0$, then $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is locally free and commutes with arbitrary base change.*

Proof. Since the question is local, we can assume that $\mathscr{L}_0^N$ is generated by global sections. In the notations from 4.2 $f_{0*}(\omega_{X_0/Y_0} \otimes \mathscr{L}_0)$ is a direct summand of $f_{1*}\omega_{X_1/Y_0}$. This sheaf however is locally free and commutes with base change, as it was shown in [1], 5.5, and in 28,4.

4.4. *The proof of* 2.7. If $v = 1$ and if f_0 is smooth, 2.7 has been shown in 3.9. In general, for $v = 1$, 2.7 is a special case of 3.24 (for $N = 1$ and $\mathscr{L} = \mathscr{O}_{X_0}$). Therefore, let us assume that $v > 1$. Using the notations from 2.6, we choose $\mathscr{H}$ to be an ample invertible sheaf on Y, $\mathscr{H}' = \delta^* \mathscr{H}$ and $\mathscr{H}_0 = j^* \mathscr{H}$.

By assumption $f_0^* f_{0*} \omega_{X_0/Y_0}^N \to \omega_{X_0/Y_0}^N$ is surjective. Replacing N by some multiple we may assume moreover that $S^b(f_{0*}\omega_{X_0/Y_0}^N) \to f_{0*}\omega_{X_0/Y_0}^{b \cdot N}$ is surjective for all $b > 0$, and that v divides N.

Let us define

$$r(\mu) = \operatorname{Min}\{s > 0; f_{0*}\omega_{X_0/Y_0}^{\mu} \otimes \mathscr{H}_0^{s \cdot \mu - 1}$$

weakly positive over Y_0 with respect to

$$(Y', f'_* \omega_{X'/Y'}^{\mu} \otimes \mathscr{H}'^{s \cdot \mu - 1})\} .$$

$r(\mu)$ is finite. In fact, we just have to use 2.8, 2, and choose some coherent extension $\mathscr{F}$ of $f_{0*}\omega_{X_0/Y_0}^{\mu}$ which is contained in $\delta_* f'_* \omega_{X'/Y'}^{\mu}$. Then $\mathscr{F} \otimes \mathscr{H}^n$ will be globally generated for $n \gg 0$. We write $r = r(N)$.

Choose $\mathscr{L}_0 = \omega_{X_0/Y_0}^{\mu - 1} \otimes f_0^* \mathscr{H}_0^{r(\mu - 1)}$ and, for some large divisor E on X' with $\operatorname{codim}_{Y'}(f'(E)) \geqq 2$, $\mathscr{L}' = \omega_{X'/Y'}^{\mu - 1} \otimes f'^* \mathscr{H}'^{r \cdot (\mu - 1)} \otimes \mathscr{O}_{X'}(E)$. For some b sufficiently large,

$$S^{(\mu - 1) \cdot b}(f_{0*}\omega_{X_0/Y_0}^N \otimes \mathscr{H}_0^{r \cdot N - 1}) \otimes \mathscr{H}_0^{(\mu - 1) \cdot b}$$

will globally generated over Y_0 by sections lying in

$$H^0(Y', S^{(\mu - 1) \cdot b}(f'_* \omega_{X'/Y'}^N \otimes \mathscr{H}'^{r \cdot N - 1}) \otimes \mathscr{H}'^{(\mu - 1) \cdot b}) .$$

Therefore the quotient sheaf

$$f_{0*}\omega_{X_0/Y_0}^{(\mu-1)\cdot b\cdot N}\otimes\mathscr{H}_0^{(\mu-1)\cdot b\cdot r\cdot N}=f_{0*}\mathscr{L}_0^{b\cdot N}$$

is generated over Y_0 by

$$H^0(Y',((f'_*\omega_{X'/Y'}^{(\mu-1)\cdot b\cdot N})\otimes\mathscr{H}'^{(\mu-1)\cdot b\cdot r\cdot N})^{**}).$$

If we choose E large enough this vector space is $H^0(X',\mathscr{L}'^{b\cdot N})$. Since

$$f_{0*}\mathscr{L}_0\otimes\omega_{X_0/Y_0}=f_{0*}\omega_{X_0/Y_0}^{\mu}\otimes\mathscr{H}_0^{r\cdot(\mu-1)}$$

we find, if ν divides μ, that all assumptions of 2.10 are satisfied for $\mathscr{L}_0$ and $\mathscr{L}'$. Therefore we obtain:

Claim 4.5. $f_{0*}(\omega_{X_0/Y_0}\otimes\mathscr{L}_0)=f_{0*}\omega_{X_0/Y_0}^{\mu}\otimes\mathscr{H}_0^{r\cdot(\mu-1)}$ is weakly positive over Y_0 with respect to $(Y',f'_*\omega_{X'/Y'}^{\mu}\otimes\mathscr{H}'^{r\cdot(\mu-1)})$.

Claim 4.6. If ν divides μ, then $(r(\mu)-1)\cdot\mu-1\leqq r(N)\ (\mu-1)$. Moreover $r(N)\leqq N$.

Proof. By the definition of $r(\mu)$ as a minimum, the sheaf

$$f_{0*}\omega_{X_0/Y_0}^{\mu}\otimes\mathscr{H}_0^{(r(\mu)-1)\cdot\mu-1}$$

will not be weakly positive over Y_0 with respect to

$$(Y',f'_*\omega_{X'/Y'}^{\mu}\otimes\mathscr{H}'^{(r(\mu)-1)\cdot\mu-1}).$$

Then however $r\cdot(\mu-1)$ must be larger than $(r(\mu)-1)\cdot\mu-1$. Since we assumed that ν divides N, we obtain $r\leqq N$.

Claim 4.7. $f_{0*}\omega_{X_0/Y_0}^{\nu}\otimes\mathscr{H}_0^{N(\nu-1)}$ is weakly positive over Y_0 with respect to

$$(Y',f'_*\omega_{X'/Y'}^{\nu}\otimes\mathscr{H}'^{N(\nu-1)}).$$

Proof. This for $r=r(N)$ and $\mu=\nu$ just following from 4.5 and 4.6.

The proof of 2.7 ends with the usual argument: 4.7 holds true for all ample sheaves $\mathscr{H}$ and, since N remains the same for all pullback families, we can apply 2.4, b, using 2.5, b, to obtain 2.7 from 4.7.

§5. Applications to fibre spaces

In [14] we considered two methods to show the quasi-projectivity of moduli spaces: one, using "Geometric Invariant Theory" and a second one, which needs however the existence of a universal family, and hence applies only to fine moduli spaces. Let us recall the second one, in a slightly more general set up.

Assumptions 5.1. Let

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{\sigma} & X' \\ \downarrow f_0 & & \downarrow f & & \downarrow f' \\ Y_0 & \xrightarrow{j} & Y & \xleftarrow{\delta} & Y' \end{array}$$

be a commutative diagram of morphisms of reduced irreducible separated analytic spaces (we assume, for simplicity, that they are irreducible, even if the same arguments work if they are of finite type). Let us assume that:

a) i and j are open embeddings, X and Y are compact, and σ and δ are desingularizations. $X - X_0$ and $Y - Y_0$ are supposed to be closed proper analytic subspaces.

b) f and f' are surjective and $X_0 = f^{-1}(Y_0)$.

c) f_0 is flat and projective and all fibres of f_0 are reduced normal varieties of dimension n with at most rational Gorenstein singularities.

d) For all fibres F of f_0, the sheaf ω_F is semi-ample and of maximal Kodaira dimension.

e) For all $y \in Y_0$, the set $\{y' \in Y_0, f^{-1}(y') \text{ birational to } f^{-1}(y)\}$ is finite.

Theorem 5.2. *Under the assumptions made in* 5.1 *write*

$$\mathscr{L}_0 = \det(f_{0*}\omega_{X_0/Y_0}^{\nu\cdot\mu})^a \otimes \det(f_{0*}\omega_{X_0/Y_0}^{\nu})^b$$

and

$$\mathscr{L}' = \det(f'_*\omega_{X'/Y'}^{\nu\cdot\mu})^a \otimes \det(f'_*\omega_{X'/Y'}^{\nu})^b .$$

Then, for $a, b, \mu, \nu \gg 0$ *the sheaf* $\mathscr{L}_0$ *is globally generated over* Y_0 *by*

$$A = H^0(Y_0, \mathscr{L}_0) \cap H^0(Y', \mathscr{L}')$$

and the induced morphism $Y_0 \to \mathbb{P}(A)$ *is an embedding.* (*Definition* 2.1, *a, makes perfectly sense for analytic spaces*).

Remarks 5.3. a) The assumption 5.1, d can be replaced by the weaker assumption discussed in 2.12, b, provided f_0 is smooth.

b) A closer look to the proof of 5.2 should reveal that, even without the assumption 5.1, e, for some values of a and b, one still finds enough sections to generate $\mathscr{L}_0$ over Y_0, provided that Y_0 is normal and ω_{X_0/Y_0} relatively ample.

c) Since 5.1, d implies that $f_{0*}\omega_{X_0/Y_0}^{\nu}$ is locally free and compatible with base change (see 2.9), one can choose the numbers ν and μ in 5.2 in the following way: for all fibres F of f_0 one wants φ_ν: $F \to \mathbb{P}(H^0(F, \omega_F^\nu))$ to be birational and the ideal of $\varphi_\nu(F)$ to be generated by homogeneous polynomials of degree μ. Moreover, the multiplication map

$$S^\mu(H^0(F, \omega_F^\nu)) \to H^0(F, \omega_F^{\nu\cdot\mu})$$

should be surjective.

In fact, one can find such ν, μ for a given F, and by base change, the same ν, μ will work for all fibres which are close by.

Before starting to sketch the proof of 5.2 it is convenient to have the following generalization of our Definition 2.2.

Definition 5.4. Let Y a reduced irreducible analytic space, i: $Y_0 \to Y$ a Zariski open dense subspace and δ: $Y' \to Y$ a desingularization. Let $\mathscr{F}_0$ be a locally free sheaf on Y_0 and $\mathscr{F}'$ a coherent reflexive sheaf on Y'. We call $\mathscr{F}_0$ *weakly positive over* Y_0 *with*

respect to $(Y', \mathscr{F}')$ if one has:

i) $\mathscr{F}_0 \subset i^* \delta_* \mathscr{F}'$

ii) Let

$$\begin{array}{ccccc} Z_0 & \xrightarrow{j} & Z & \xleftarrow{\sigma} & Z' \\ {\scriptstyle \alpha_0}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha'}\downarrow \\ Y_0 & \xrightarrow{i} & Y & \xleftarrow{\delta} & Y' \end{array}$$

be a commutative diagram, where $Z_0 = \alpha_0^{-1}(Y_0)$, Z is projective and σ a desingularization. Then $\alpha_0^* \mathscr{F}_0$ is weakly positive over Z_0 with respect to $(Z', \alpha'^* \mathscr{F}'/\text{torsion})$.

Remark 5.5. It is likely, that this definition is not the "right" one, if Y is not Moišezon. However, in all cases where we use it, it will turn out later, that Y was Moišezon.

Proof of 5.2. Since by 2.9 the sheaves $f_{0*} \omega^{\eta}_{X_0/Y_0}$ are locally free and compatible with base change, for $\eta > 1$, they are weakly positive over Y_0 with respect to $(Y', f'_* \omega^{\eta}_{X'/Y'})$ by 2.7. Then, if we choose ν and μ as in 5.3, c, the multiplication map $S^{\mu}(f_{0*} \omega^{\nu}_{X_0/Y_0}) \to f_{0*} \omega^{\nu \cdot \mu}_{X_0/Y_0}$ is surjective and its kernel in $y \in Y_0$ determines the birational equivalence class of $f_0^{-1}(y)$. Moreover, by 2.9, the sheaf $f_{0*} \omega^{\nu}_{X_0/Y_0}$ is weakly positive over Y_0 with respect to $(Y', f'_* \omega^{\nu}_{X'/Y'})$. One can finish the proof as we did for 1.18 in [14]. However, we want to reformulate the principle behind the arguments used there.

In fact, as J. Kollár pointed out, there is hidden in [14] some quite general ampleness criterion. It follows from arguments similar to those used in [14], §2 and §4, however it was not stated there. We use the pretext of 5.2 to formulate it:

5.6. For Y_0, Y and Y' as in 5.4 and $i = 1, \ldots, s$, let $\mathscr{F}_0^{(i)}$ be locally free sheaves on Y_0, $\mathscr{F}'^{(i)}$ coherent reflexive sheaves on Y', such that $\mathscr{F}_0^{(i)}$ is weakly positive over Y_0 with respect to $(Y', \mathscr{F}'^{(i)})$. Let $r_i = \operatorname{rank}(\mathscr{F}_0^{(i)}) = \operatorname{rank}(\mathscr{F}'^{(i)})$ and let T_i: $Gl(r_i, \mathbb{C}) \to V_i$ be finite dimensional positive representations ([14] §2 or [8] §3). As we did in §2 of [14] we can consider the induced tensor bundles $T_i(\mathscr{F}_0^{(i)})$ and "taking reflexive hulls" $T_i(\mathscr{F}'^{(i)})$. Let Q_0 be a locally free sheaf on Y_0, Q' a coherent reflexive sheaf on Y' with $Q_0 \subset i^* \delta_* Q'$. Moreover, assume we have a surjective map

$$m_0 \colon \bigotimes_{i=1}^{s} T_i(\mathscr{F}_0^{(i)}) \to Q_0$$

and a map

$$m' \colon \bigotimes_{i=1}^{s} T_i(\mathscr{F}'^{(i)}) \to Q'$$

which are compatible with each other under $i^* \delta_*$. For $y \in Y_0$ the kernel of m_0 in y is a $k = \operatorname{rank}(\bigotimes_{i=1}^{s} T_i(\mathscr{F}_0^{(i)})) - \operatorname{rank}(Q_0)$ dimensional subspace of $\bigotimes_{i=1}^{s} T_i(\mathbb{C}^{r_i})$. For $G = \times_{i=1}^{s} \mathbb{P}Gl(r_i, \mathbb{C})$ we obtain thereby a G orbit $G \cdot y$ in the Grassmannian $\operatorname{Gr}(k, \bigotimes_{i=1}^{s} T_i(\mathbb{C}^{r_i}))$.

Ampleness Criterion 5.7. *Under the assumptions made in* 5.6, *assume that for all* $y \in Y_0$ *the set* $\{y' \in Y_0;\ G \cdot y = G \cdot y'\}$ *is finite and moreover* $\dim(G \cdot y) = \dim(G)$. *Then, for* $a, b_1, \ldots, b_s \gg 0$, *the sheaf* $\mathscr{L}_0 = \det(Q_0)^a \otimes \bigotimes_{i=1}^s \det(\mathscr{F}_0^{(i)})^{b_i}$ *is globally generated over* Y_0 *by*

$$A = H^0(Y_0, \mathscr{L}_0) \cap H^0(Y', \det(Q')^a \otimes \bigotimes_{i=1}^s \det(\mathscr{F}'^{(i)})^{b_i})$$

and the induced morphism $Y_0 \to \mathbb{P}(A)$ *is an embedding.*

Remark 5.8. a) Obviously the multiplication map considered in the proof of 5.2 satisfies for $s = 1$, all the assumptions made in 5.6 and 5.7.
b) J. Kollár replaced in [8], §3, the descend argument from [14], 2.7 by some kind of Nakai-Moišezon criterion. One advantage is that he obtains $b_i = 0$ in 5.7, the disadvantage is that Y_0 must be compact in his case.
c) 5.7 holds true without "with respect to $(Y', \ldots)$", in the set up of "weak positivity over Y_0". The proof remains the same.

Sketch of the proof of 5.7. As in 2.3 we may replace Y by $\mathrm{Spec}_Y(\mathscr{A}_Y)$ and assume that $\mathscr{A}_Y = \mathscr{O}_Y$. Let us write $\mathscr{F}^{(i)} = \delta_* \mathscr{F}'^{(i)} \cap i_* \mathscr{F}_0^{(i)}$ and $Q = \delta_* Q' \cap i_* Q_0$. Blowing up Y we can assume that $\mathscr{F}^{(i)}$ and Q are locally free and that the induced map $m\colon \bigotimes_{i=1}^s T_i(\mathscr{F}^{(i)}) \to Q$ is surjective. For $\mathscr{E}_i = \bigoplus^{r_i} \mathscr{F}^{(i)}$ we consider

$$\mathbb{P}_i = \mathbb{P}(\mathscr{E}_i^\vee) \xrightarrow{\pi_i} Y$$

$$\mathbb{P} = \mathbb{P}_1 \times_Y \ldots \times_Y \mathbb{P}_s \xrightarrow{\pi} Y$$

and $\mathbb{P}_0 = \pi^{-1}(Y_0)$. The maps $\pi_i^* \mathscr{E}_i^\vee \to \mathscr{O}_{\mathbb{P}_i}(1)$ induce "universal bases"

$$s_i\colon \bigoplus^{r_i} \mathrm{pr}_i^* \mathscr{O}_{\mathbb{P}_i}(-1) \to \pi^* F^{(i)}$$

and

$$s = \bigoplus_{i=1}^s s_i\colon \bigoplus_{i=1}^s (\bigoplus^{r_i} \mathrm{pr}_i^* \mathscr{O}_{\mathbb{P}_i}(-1)) \to \bigoplus_{i=1}^s \pi^* \mathscr{F}^{(i)}\,.$$

For $\eta \in \mathbb{Z}$, let us denote

$$\mathscr{O}_{\mathbb{P}}(\eta \cdot \underline{r}) = \left(\bigotimes_{i=1}^s \mathrm{pr}_i^* \mathscr{O}_{\mathbb{P}_i}(r_i) \right)^\eta \quad \text{and} \quad \mathscr{F} = \bigoplus_{i=1}^s \mathscr{F}^{(i)}\,.$$

If D is the degeneration divisor of s, then $\mathscr{O}_{\mathbb{P}}(D) = \pi^* \det(\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}}(\underline{r})$. As in 2.6 of [14] one obtains:

The natural inclusion

$$\rho^\eta\colon \mathscr{O}_Y \to \pi^* \mathscr{O}_{\mathbb{P}}(\eta \cdot D) = \det(\mathscr{F})^\eta \otimes \mathscr{O}_{\mathbb{P}}(\eta \cdot \underline{r})$$

splits.

In fact ρ is induced by the natural map

$$\det(\mathscr{F})^{-1} = \bigotimes_{i=1}^s \det(\mathscr{F}^{(i)})^{-1} \to \bigotimes_{i=1}^s (\bigotimes^{r_i} \mathscr{F}^{(i)\vee}) \to \bigotimes_{i=1}^s S^{r_i}(\mathscr{E}_i^\vee) = \pi_* \mathscr{O}_{\mathbb{P}}(\underline{r})\,.$$

On $\mathbb{P}$ we have the induced map

$$M\colon \bigotimes_{i=1}^s \mathrm{pr}_i^* T_i(\bigoplus^{r_i} \mathscr{O}_{\mathbb{P}}(-1)) \to \bigotimes_{i=1}^s T_i(\pi^* \mathscr{F}^{(i)}) \to \pi^* Q\,.$$

and M is surjective over $\mathbb{P}_0 - D$. Let

$$h\colon \mathbb{P}_0 - D \to \mathrm{Gr}\left(k, \bigotimes_{i=1}^{s} T_i(\mathbb{C}^{r_i})\right) \subseteq \mathbb{P}^M$$

be the induced map where the right hand inclusion is given by the Plücker coordinates. By assumption h is quasi-finite.

Let $\tau\colon \mathbb{P}' \to \mathbb{P}$ be a blowing up with centers outside $\mathbb{P}_0 - D$, such that $h' = h_0 \circ \tau\colon \mathbb{P}' \to \mathbb{P}^M$ is a morphism, and such that for some $a > 0$ $h'^* \mathcal{O}_{\mathbb{P}^M}(a)$ contains an ample sheaf $\mathscr{H}$, isomorphic to $h'^* \mathcal{O}_{\mathbb{P}^M}(a)$ over $\mathbb{P}_0 - D$. Write $\pi' = \pi \circ \tau\colon \mathbb{P}' \to Y$.

Since $h^* \mathcal{O}_{\mathbb{P}^M}(1) = \pi^*(\det(Q) \otimes \det(\bigotimes_{i=1}^{s} T_i(\mathscr{F}^{(i)})^{-1})$ we can find some divisor D', supported in $\tau^{-1}(D)$, an effective divisor E, supported in $\tau^{-1}(\mathbb{P} - \mathbb{P}_0)$, and some $b_i' \in \mathbb{Z}$ such that

$$\pi'^*(\det(Q)^a \otimes \bigotimes_{i=1}^{s} \det(F^{(i)})^{b_i}) \otimes \mathcal{O}_{\mathbb{P}'}(D' - E)$$

is ample.

The assumptions made on $\mathscr{F}'^{(i)}$ and $\mathscr{F}_0^{(i)}$, the choice of Y and 2.3, b imply that $\pi'^* \mathscr{F}$ is weakly positive over $\pi'^{-1}(\mathbb{P}_0)$. Then as in 4.7 of [14] one finds $\mathcal{O}_{\mathbb{P}'}(\tau^* D) \otimes \bigotimes_{i=1}^{s} \pi'^* \det(\mathscr{F}^{(i)})^{r_i^{-1}}$ to have the same property. Tensorizing with a high power of this sheaf, we can, for $b_i \gg 0$, force D' to become effective. Then, 2.7 of [14] allows to "descend sections to Y" and to prove 5.7.

Remark 5.9. Let $\mathscr{M}_h$ be one of the moduli functors considered in 0.2. Then there exists an analytic coarse moduli space M_h^{an} for $\mathscr{M}_h$. One can use 5.2 to show that the normalization $M_h^{\sim}$ of M_h^{an} is quasi-projective:

In fact, by [8] §2, we know inbetween that over some finite cover $\tau_0\colon Y_0 \to M_h^{an}$ there exists some "universal family" $f\colon X_0 \to Y_0$. By 5.2 we find an ample sheaf on Y_0, which is the pullback of some invertible sheaf on M_h^{an}.

However, as long as M_h^{an} is not normal, the property of ampleness is not compatible with finite coverings. Therefore, in this way we only obtain:

a) $M_h^{\sim}$ is quasi-projective

b) If the non normal locus of M_h^{an} is compact, then M_h^{an} is quasi-projective.

However, the only obstruction to obtain 0.3 from 5.2 seems to be, that we do not know whether we can choose $\tau_0\colon Y_0 \to M_h^{an}$ in such a way that the trace map from $\tau_{0*} \mathcal{O}_{Y_0}$ has its image in $\mathcal{O}_{M_h^{an}}$. May be, a closer look to the arguments used in [8] §2 would settle this problem. Since [8] only considers compact moduli spaces (for more general moduli problems), the assumption made in b holds there.

§6. Applications to moduli problems

Fortunately we formulated 1.7, b in [14] as well as its prove in such a way, that 2.9 implies theorem 0.3. Let us nevertheless make some comments on 0.3, [14] 1.7, b, and their proves.

The assumptions on the moduli functors $\mathcal{M}_h$ in 0.2 are needed in order to have "nice Hilbert schemes". Let us state what we need exactly:

Assumptions 6.1. Let $h(T)$ be a polynomial of degree n and $\mathcal{M}'_h$ the moduli functor of complex projective normal irreducible varieties F of dimension n with at most rational Gorenstein singularities, such that ω_F is semi-ample, of Kodaira dimension n and such that for $\nu > 0$ one has $h(\nu) = \chi(F, \omega_F^\nu)$.

Let $\mathcal{M}_h$ be a subfunctor of $\mathcal{M}'_h$ such that for some number $\nu > 1$ one has:

i) For all $F \in \mathcal{M}_h(\mathbb{C})$ the sheaf ω_F^ν is generated by its global sections and for the induced map $\varphi_\nu\colon F \to \mathbb{P}(H^0(F, \omega_F^\nu))$, the image $\varphi_\nu(F)$ is normal and birational to F.

ii) There exists some separated scheme H of finite type over $\mathbb{C}$ and a universal family $h\colon \mathcal{X} \to H \in \mathcal{M}_h(H)$ together with an isomorphism

$$\Phi_\nu\colon \mathbb{P}(h_*\omega^\nu_{\mathcal{X}/H}) \cong \mathbb{P}^{r-1} \times H$$

over H such that (h, Φ_ν) is universal, i.e. for all $f\colon X \to Y \in \mathcal{M}_h(Y)$ and $\rho_\nu\colon \mathbb{P}(f_*\omega^\nu_{X/Y}) = \mathbb{P}^{r-1} \times Y$ there is a unique morphism $Y \to H$ such that (f, ρ_ν) is obtained as pullback of (h, Φ_ν).

iii) The action of $G = \mathbb{P}Gl(r, \mathbb{C})$ on H obtained by "change of coordinates in $\mathbb{P}^{r-1}$" is proper.

Let us write $\lambda_\eta = \det(h_*\omega^\eta_{\mathcal{X}/H})$ and $r(\eta) = \operatorname{rank}(h_*\omega^\eta_{\mathcal{X}/H})$. Obviously for all η the sheaf λ_η is G-linearized (see [10] Def. 1.6). Recall that we denote by $H(\mathcal{L})^s$ the stable points of H with respect to $\mathcal{L}$ and under the G-action (see [10], Def. 1.7 and [14], 5.2).

Then the main result of [14] and of this paper is

Theorem 6.2. *Under the assumptions made in* 6.1 *one has*

a) *For $a, b, \mu \gg 0$ one has $H = H(\lambda^a_{\nu\cdot\mu} \otimes \lambda^b_\nu)^s$.*
b) *There exists a coarse quasi-projective moduli scheme M_h for $\mathcal{M}_h$*
c) *$\lambda^a_{\nu\cdot\mu} \otimes \lambda^b_\nu$ descends for $a, b, \mu \gg 0$ to an ample invertible sheaf on M_h.*

Remark 6.3. It seems that, using the assumptions mentioned in 2.12, b, one can as well consider $\mathcal{M}'_h$ to be the moduli functor of complex projective manifolds F such that for some fixed N one finds an invertible semi-ample subsheaf $\mathcal{P}$ of ω_F^N of maximal Iitaka dimension and

$$\dim H^0(F, \mathcal{P}^\nu) = \dim H^0(F, \omega_F^{\nu\cdot N}) = h(\nu\cdot N)\,.$$

However, one would need that $\mathcal{P}$ behaves well on $\mathcal{X} \to H$.

Remark 6.4. If $\mathcal{M}_h$ is the moduli functor of curves or surfaces of general type, then D. Mumford [10] and D. Gieseker [4] proved 6.2. However, they were able to take $b = -\dfrac{r(\nu\cdot\mu)\cdot\mu}{r(\nu)}\cdot a$. Since the sheaf corresponding to λ_ν is weakly positive over M_h their result implies ours for $n \leqq 2$, but not vice versa.

Some remarks on the proof of 2.9 *and on the proof of* "2.9 $\Rightarrow$ 6.2" *in* [14], §5:

6.5. The starting point is 2.9. If $\mathcal{M}_h$ is the moduli functor of canonically polarized manifolds with Hilbert polynomial h, then 2.9 is just needed for smooth morphisms $f_0\colon X_0 \to Y_0$. In this case the part "3.10–4.1" of this paper is unnecessary.

If $\mathcal{M}_h$ is the moduli functor of surfaces of general type, then again it would be sufficient to know 2.9 for smooth morphisms, provided that for all families $f_0\colon X_0 \to Y_0 \in \mathcal{M}_h(Y_0)$ of normal surfaces with rational double points, one has "simultaneous resolution". That means that one can find $\tau_0\colon X_0' \to X_0$ such that all fibres of $f_0' = f_0 \circ \tau_0$ are minimal desingularizations of the corresponding fibres of f_0. However, I do not know whether this can be done.

6.6. The second part, "how to use 2.9 in order to get 6.2" is completely contained in §5 of [14]. The proof given there uses only in 5.9 the special structure of the universal family and the prove of 5.9 in 5.14 of [14] works as well and without any change under the assumptions of 6.2. The sections §2–§4 of [14] are no longer necessary in the proof of 6.2. The properties, recalled in 2.4, and 2.9 are enough.

Comments and open problems

6.7. It might be possible to extend the results of this paper to moduli problems of multicanonically polarized varieties with canonically singularities, at least if one has a reasonable Hilbert scheme and if one knows that any deformation of those objects over a non singular base has canonical singularities in the total space. We leave it to the reader to estimate the technical difficulties one could run into, trying to generalize §3 to this case.

6.8. In principle the methods from [14] together with 2.10 should enable us to construct quasi-projective moduli spaces for arbitrary polarized compact complex manifolds, let us say with trivial Pic^0. However one has to find a natural polarization $\mathcal{L}$ on the total space X for all $f\colon X \to Y \in \mathcal{M}_h(Y)$, such that $f_*\mathcal{L}$ is weakly positive over Y. Since $\mathcal{L}$ is only determined up to $f^*(\mathrm{Pic}(Y))$ we can always force $f_*\mathcal{L}$ to be weakly positive, however in order to apply the methods from §5 of [14] we have to do it in a functorial way. It seems that this can be done and we hope to be able to use this construction to prove in a third part of this paper the existence of quasi-projective moduli schemes for bounded and separated moduli functors of polarized normal varieties with rational Gorenstein singularities and irregularity zero, whose canonical sheaves are numerically effective.

6.9. And what about moduli of vector bundles? Again the problem seems to be to find a good universal bundle to start with. The only case I know, where an approach similar to [14] was applied successfully, is J. Kollár's proof of the projectivity of the compactified Picard scheme in [8], §6. May be, his construction of a rigidification of the Poincaré bundle and his method to apply the ampleness criterion [8], 3.9 (see also 5.7) could give some hint, how to attack moduli of vector bundles.

6.10. As the reader has seen, we needed at some point some extension of the sheaves considered to a compactification, because 2.4, c, was the only way to descend "weak positivity" to singular varieties. It would have been more natural to this aim, to look for "good compactifications of morphisms" instead of "Schmid extensions" of direct images of dualizing sheaves. May be, one can use methods from §1 to construct those.

6.11. Of course, the existence of good compactifications of morphisms would also follow from compactifications of the moduli functors $\mathscr{M}_h$. Since in Theorem 0.3 and 6.2 we even know that the sections of $\lambda_{\nu \cdot \mu}^a \otimes \lambda_\nu^b$ which embedd M_h in a projective space are coming from some compactification of a desingularization H, it seems possible that some of the methods used in this paper could be of some help, if one wants to construct compactifications $\bar{\mathscr{M}}_h$ of $\mathscr{M}_h$.

References

1. Deligne, P.: Equations différentielles à points singuliers réguliers. (Lect. Notes Math., Vol. 163). Berlin-Heidelberg-New York: Springer 1970
2. Deligne, P.: Théorèmes de Lefschetz et critères de dégénérecence de suites spectrales. Publ. Math., Inst. Hautes Etud. Sci. **35**, 107–126 (1968)
3. Esnault, H., Viehweg, E.: Logarithmic De Rham complexes and vanishing theorems. Invent. Math. **86**, 161–194 (1986)
4. Gieseker, D.: Global moduli for surfaces of general type. Invent. Math. **43**, 233–282 (1977)
5. Kawamata, Y.: Characterization of abelian varieties. Compos. Math. **43**, 253–276 (1981)
6. Kollár, J.: Higher direct images of dualizing sheaves. Ann. Math. **123**, 11–42 (1986)
7. Kollár, J.: Subadditivity of the Kodaira dimension: Fibres of general type. Proc. Sympos. Alg. Geom. Sendai 1985. (Adv. Stud. Pure Math., Vol. 10, pp 361–398). North-Holland 1987
8. Kollár, J.: Projectivity of complete moduli. Differ. Geom. (in press)
9. Mori, S.: Classification of higher-dimensional varieties. Algebraic Geometry. Bowdoin 1985. Proc. Symp. Pure Math. **46**, 269–331 (1987)
10. Mumford, D., Fogarty, J.: Geometric Invariant Theory, Second Edition. (Ergebnisse der Math., Vol. 34). Berlin-Heidelberg-New York: Springer 1982
11. Schmid, W.: Variations of Hodge structures: the singularities of the period mapping. Invent. Math. **22**, 211–319 (1973)
12. Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. Alg. Varieties and Anal. Varieties. (Adv. Stud. Pure Math. Vol. 1, pp. 329–353). North-Holland 1983
13. Viehweg, E.: Vanishing theorems and positivity in algebraic fibre spaces. Proc. of the Intern. Congr. of Math. Berkeley 1986, 682–688
14. Viehweg, E.: Weak positivity and the stability of certain Hilbert points. Invent. Math. **96**, 639–667 (1989)

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Cyclicity and generation of points mod p on elliptic curves

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1. Introduction

In this paper we study the group of points modulo p of elliptic curves defined over $\mathbb{Q}$. In particular, we are interested in the frequency with which this group is cyclic and with which it is generated by a fixed set of global rational points. Let E be an elliptic curve over $\mathbb{Q}$ and for each prime p where E has good reduction, let $\bar{E}(\mathbb{F}_p)$ be the group of rational points on the reduction of E modulo p. In [14], Serre raised the question of how often this group is cyclic, and following Hooley's work [6] on Artin's primitive root conjecture, showed that the number of such $p \leqq x$ is $\sim cx/\log x$ for some constant c, assuming the Generalized Riemann Hypothesis (GRH). The second author removed this hypothesis for curves with complex multiplication (CM) in [9], and also demonstrated unconditionally the existence of infinitely many primes for which $\bar{E}(\mathbb{F}_p)$ is cyclic in [10], for certain non-CM elliptic curves. Though the method of [9] establishes an asymptotic formula, the method of [10] does not give a good lower bound for the number of such primes. We will prove unconditionally:

Theorem 1. *The group $\bar{E}(\mathbb{F}_p)$ is cyclic for infinitely many p if and only if E has a non-rational 2-division point, and moreover, in this case, the number of primes $p \leqq x$ for which $\bar{E}(\mathbb{F}_p)$ is cyclic is $\gg x/\log^2 x$.*

The proof of this result is given in section 2; the key is a result from the theory of sieves.

In [3] we discussed the following question, raised by Lang and Trotter [8]: Given a rational point X of infinite order in the Mordell-Weil group $E(\mathbb{Q})$, how often does the reduction of X mod p equal the full group $\bar{E}(\mathbb{F}_p)$? This question is analogous to the classical question of how often a given integer is a primitive root mod p, but the situation cannot be handled by Hooley's method, owing to the large

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size of the conjugacy classes. However, if E has CM by the ring of integers of the imaginary quadratic field k, the authors were able to prove in [3] that the set of primes p such that $\bar{E}(\mathbb{F}_p) = \langle \bar{X} \rangle$ and p splits in k has a density. This method does not apply to primes which are inert in k, and we must consider the following somewhat weaker question, also raised in [8].

Let Γ be a free subgroup of $E(\mathbb{Q})$ of rank $r(\Gamma)$ and define

$$S(\Gamma) = \{p \text{ prime} | \bar{E}(\mathbb{F}_p) = \Gamma_p\},$$

where Γ_p is the reduction of Γ mod p. Also, if E has CM by k, let

$$S'(\Gamma) = \{p \text{ prime} | \bar{E}(\mathbb{F}_p) = \Gamma_p \text{ and } p \text{ is inert in } k\}.$$

One expects the sets $S(\Gamma)$ and $S'(\Gamma)$ to have certain densities $\delta(\Gamma)$ and $\delta'(\Gamma)$ (see sections 4 and 5). Provided $r(\Gamma)$ is sufficiently large, one can verify these expectations under the assumption of the GRH (see [3]). The ranks for which our results apply can be lowered substantially by not insisting on producing the full densities above but rather positive multiples of them. Such results were previously known (see [3]) only for CM curves with $r(\Gamma) \geqq 10$ and non-CM curves with $r(\Gamma) \geqq 18$.

Theorem 2. *If E has CM and $r(\Gamma) \geqq 2$, then, under the GRH, $S'(\Gamma)$ contains a set of density $\geqq (1 - \log 2 - \varepsilon)\delta'(\Gamma)$ for any $\varepsilon > 0$*

Theorem 3. *If E has CM and $r(\Gamma) \geqq 6$, then, under the GRH, $S(\Gamma)$ has density $\delta(\Gamma)$.*

Theorem 4. *If E does not have CM and $r(\Gamma) \geqq 7$, then, under the GRH, for every $\varepsilon > 0$, $S(\Gamma)$ contains a set of density*

$$(1 - \log(20/9) - \varepsilon)\delta(\Gamma),$$

and if we assume Artin's holomorphy conjecture, then $S(\Gamma)$ contains a set of density

$$(1 - \log(4/3) - \varepsilon)\delta(\Gamma).$$

The holomorphy conjecture referred to above is the statement that all Artin L-series of the extensions $L_n/\mathbb{Q}$ are analytic at $s \neq 1$, where the fields L_n are as defined in section 3. Theorems 2 and 3 rely on the improved error estimate in the Chebotarev density theorem of [12], and this improvement is conditional on Artin's holomorphy conjecture. Section 3 is devoted to showing that this holds for curves with CM. In section 4 we give the proofs of Theorems 2 and 3, and in section 5 we prove theorem 4.

2. Cyclicity of $\bar{E}(\mathbb{F}_p)$

We let E be a fixed elliptic curve over $\mathbb{Q}$, and let $K_n = \mathbb{Q}(E[n])$ for each positive integer n, where $E[n]$ is the set of n-division points of E. If p is a prime of good reduction, it is easy to see that for any prime q, $(\mathbb{Z}/q\mathbb{Z})^2 \subset \bar{E}(\mathbb{F}_p)$ if and only if p splits completely in K_q. This gives:

Lemma 1. *If p is a prime of good reduction, $\bar{E}(\mathbb{F}_p)$ is cyclic if and only if p does not split completely in K_q for any prime q.*

On the GRH, primes which do not split completely in any K_q can be successfully enumerated, and an inclusion-exclusion argument leads to the following result due to Serre [14] (see also [9]).

Theorem. *Assuming GRH, the primes p for which $\bar{E}(\mathbb{F}_p)$ is cyclic have density*

$$\delta = \sum_{n \geq 1} \mu(n)/[K_n:\mathbb{Q}] ,$$

where $\mu(n)$ denotes the Möbius function.

We recall that Serre [13] has shown that for E without CM,

$$[K_n:\mathbb{Q}] \asymp |GL_2(\mathbb{Z}/n\mathbb{Z})| ,$$

and if E has CM by $\mathcal{O}_k$, then

$$[K_n:\mathbb{Q}] \asymp |(\mathcal{O}_k/n\mathcal{O}_k)^x| ,$$

so that the infinite sum above converges absolutely.

If any of the fields K_q for q prime is trivial, then clearly $\delta = 0$, and by the following well-known fact, this can only happen for $q = 2$.

Lemma 2. *The cyclotomic field $\mathbb{Q}(\zeta_n)$ is contained in K_n for any n.*

Proof. This follows from properties of the Weil pairing e_n, a non-degenerate bilinear pairing $e_n: E[n] \times E[n] \to \mu_n$, where μ_n is the group of n-th roots of unity. The non-degeneracy of e_n gives its surjectivity, and its Galois invariance means that its image is contained in K_n. See [16, Cor 8.1.1 p. 98] for details.

One can show that δ vanishes exactly when $K_2 = \mathbb{Q}$, and this gives in particular the infinitude of the set of primes p for which $\bar{E}(\mathbb{F}_p)$ is cyclic if $K_2 \neq \mathbb{Q}$, on GRH. The key to eliminating GRH in Theorem 1 is the following lemma from sieve theory.

Lemma 3. *Let $S_\varepsilon(x)$ be the set of primes $p \leqq x$ such that all odd prime divisors of $p - 1$ are distinct and $\geqq x^{1/4+\varepsilon}$, p does not split completely in the field K_2, and E has good reduction at p. Then if $K_2 \neq \mathbb{Q}$ there is an $\varepsilon > 0$ such that $|S_\varepsilon(x)| \gg x/\log^2 x$.*

Proof. Since K_2 contains a non-trivial abelian extension of $\mathbb{Q}$ if it is non-trivial (it is the splitting field of a cubic polynomial), the restriction that p not split completely in K_2 can be insured by imposing a congruence condition on p. This condition can be added to the lower bound sieve of, for example, Fouvry and Iwaniec [1] (see [2], [5]) to give the desired lower bound.

Proof of Theorem 1. Fix an ε satisfying Lemma 3 and define

$$S(a, x) = \{p \in S_\varepsilon(x) | a_p = a\}$$

for each integer a with $|a| \leqq 2x^{1/2}$, where a_p denotes the trace of the Frobenius of E at p. By Weil's Theorem, $S_\varepsilon(x)$ is the (disjoint) union of these $S(a, x)$. Now for each $S(a, x)$, we enumerate those $p \in S(a, x)$ for which $\bar{E}(\mathbb{F}_p)$ is not cyclic. Indeed, if $\bar{E}(\mathbb{F}_p)$ is not cyclic, then $(\mathbb{Z}/q\mathbb{Z})^2 \subset \bar{E}(\mathbb{F}_p)$ for some prime q, and by Lemma 1 and the

definition of $S_\varepsilon(x)$, q is odd and p splits completely in K_q. By Lemma 2, p also splits completely in $\mathbb{Q}(\zeta_q)$, so we have $p \equiv 1 \bmod q$, i.e.

$$q|p-1\,. \tag{1}$$

Moreover, $|\bar{E}(\mathbb{F}_p)| = p+1-a$, so $q^2|p+1-a$ and thus $q|a-2$. Notice that a cannot equal 2 since then q^2 would divide $p-1$. Now by (1), $q \geqq x^{1/4+\varepsilon}$ and since $|a-2| \ll x^{1/2}$, q is determined by a, for x sufficiently large. Any $p \in S(a,x)$ for which $\bar{E}(\mathbb{F}_p)$ is not cyclic satisfies

$$p \equiv a-1 \bmod q^2$$

and the number of such p is

$$< x/q^2 + O(1) \ll x^{1/2-2\varepsilon}\,.$$

The total number of $p \in S_\varepsilon(x)$ for which $\bar{E}(\mathbb{F}_p)$ is not cyclic is therefore

$$\ll x^{1/2-2\varepsilon}x^{1/2} = o(x/\log^2 x)\,,$$

and this completes the proof.

3. Some group-theoretic preliminaries

Throughout this section we assume that E has CM by the ring of integers of k and let Γ be a free subgroup of $E(\mathbb{Q})$ of rank $r(\Gamma)$. For each positive integer $n \geqq 3$, let $L_n = \mathbb{Q}(E[n], \frac{1}{n}\Gamma)$, a Galois extension of $\mathbb{Q}$ with Galois group over $\mathbb{Q}$ equal to a semidirect product of a subgroup of $GL_2(\mathbb{Z}/n\mathbb{Z})$ and a subgroup of $E[n]^{r(\Gamma)}$. Also, set $L_2 = k\mathbb{Q}(E[2], \frac{1}{2}\Gamma)$ and $G_2 = \mathrm{Gal}(L_2/\mathbb{Q})$; note that by Lemma 6 of [9], $k \subset L_n$ for all n. In this section we will prove the following key fact for the proof of Theorems 2 and 3.

Proposition 1. *Artin's holomorphy conjecture holds for the extensions $L_n/\mathbb{Q}$.*

Before proving this, we need some preliminary facts about characters of semidirect products. Let A and H be two subgroups of a finite group G and suppose that A is normal and $G = AH$ with $A \cap H = 1$. That is, G is a semidirect product of H by A. Suppose in addition that A is abelian. In such a situation, it is classical knowledge that all the irreducible representations of G can be constructed from those of H (see e.g. Serre [15]).

Indeed, if χ is a character of A and $h \in H$, define χ^h by

$$\chi^h(a) = \chi(h^{-1}ah)\,.$$

Then χ^h is a character of A and this gives an action of H on the character group of A (which is isomorphic to A since A is abelian). Let H_χ be the stabilizer of χ and extend χ to $G_\chi = AH_\chi$ by defining $\tilde{\chi}(ah) = \chi(a)$. Then $\tilde{\chi}$ is still a one-dimensional character of G_χ. Now let ρ be any irreducible representation of H_χ; we can view ρ as a representation of G_χ by factoring through A. Set

$$\psi_{\chi,\rho} = \mathrm{Ind}_{G_\chi}^{G}(\tilde{\chi} \otimes \rho)\,.$$

An easy computation shows that $(\psi_{\chi,\rho}, \psi_{\chi,\rho}) = 1$ and therefore $\psi_{\chi,\rho}$ is irreducible. Moreover, all irreducible representations of G arise as one of these.

Proof of Proposition 1. We suppose that $n \geqq 3$, but the same argument as below, with minor modifications, works for L_2 as well. Let $K_n = \mathbb{Q}(E[n])$ and denote by M the Galois group of $K_n/\mathbb{Q}$. Let H be the Galois group of L_n/K_n and write G for the Galois group of $L_n/\mathbb{Q}$. Every irreducible character of G is obtained as follows. Take $\chi \in \hat{H}$, the character group of H, which we know is abelian. Then if $G_\chi = HM_\chi$, every irreducible representation is of the form

$$\psi_{\chi,\rho} = \mathrm{Ind}_{G_\chi}^{G}(\tilde{\chi} \otimes \rho) .$$

Suppose that M_χ is not abelian (for otherwise, the L-series attached to $\psi_{\chi,\rho}$ is entire). Let $M_0 = \mathrm{Gal}(K/k)$ where k is the CM field of E; we know that M_0 is abelian. Then we have an exact sequence

$$1 \to M_0 \to M \to \{\pm 1\} \to 1$$

and as $M_\chi \subset M$ and M_χ is not abelian, there is an element $\sigma_\chi \in M_\chi$ such that $\sigma_\chi \mapsto -1$. We can view elements of M as ordered pairs (τ, σ) where $\tau = \pm 1$ and $\sigma \in M_0$. With this notation, write $\sigma_\chi = (-1, \varepsilon)$. Then

$$\sigma_\chi^2 = (-1, \varepsilon)(-1, \varepsilon) = (1, -\varepsilon + \varepsilon) = (1, 0) .$$

If $g \in M_\chi \subset M$, then $g = (\tau, m_0)$. If $\tau = 1$, then $g \in M_0$. If $\tau = -1$,

$$g\sigma_\chi = (-1, m_0)(-1, \varepsilon) = (1, -\varepsilon + m_0) \in M_\chi \cap M_0 .$$

Hence every element of M_χ can be written as $(M_\chi \cap M_0)\tau^j, j = 0$ or 1. We have an exact sequence

$$1 \to M_\chi \cap M_0 \to M_\chi \to \{\pm 1\} \to 1 .$$

Thus, ρ can be written as

$$\rho = \mathrm{Ind}_{M_\chi'}^{M_\chi} \omega$$

for some abelian character ω of a subgroup $M_\chi' \subset M_\chi$. Since

$$\mathrm{Ind}_{M_\chi'}^{M_\chi} \omega = \mathrm{Ind}_{HM_\chi'}^{HM_\chi} \omega ,$$

we have

$$\tilde{\chi} \otimes \mathrm{Ind}_{HM_\chi'}^{HM_\chi} \omega = \mathrm{Ind}_{HM_\chi'}^{HM_\chi}(\omega \otimes \tilde{\chi}|_{HM_\chi'})$$

by Frobenius reciprocity. Therefore, every irreducible representation of G is of the form

$$\mathrm{Ind}_{G_\chi}^{G}(\tilde{\chi} \otimes \rho) = \mathrm{Ind}_{G_\chi}^{G}(\mathrm{Ind}_{G_\chi'}^{G_\chi}(\omega \otimes \tilde{\chi})) = \mathrm{Ind}_{G_\chi'}^{G}(\omega \otimes \tilde{\chi})$$

where $G_\chi' = HM_\chi'$. Thus every character of G is monomial and Artin's conjecture holds.

4. Generation of $\bar{E}(\mathbf{F}_p)$ in the CM case

We keep the notation and assumptions of section 3 and set $G_n = \mathrm{Gal}(L_n/\mathbb{Q})$. We were able to show in [3] that if Γ has rank 1, the reduction of Γ generates the group

$\bar{E}(\mathbb{F}_p)$ for a density α of primes p which split in k, on the GRH. The argument follows Hooley's [6] once one notes that the condition "q divides the index of Γ in $\bar{E}(\mathbb{F}_p)$" can be expressed in terms of the splitting completely of a prime π in k above p in certain extensions of k. The number of these primes can be sufficiently well estimated using the result of Lagarias and Odlyzko [7] to prove that they have a density. This method falls through if p is inert in k since the divisibility condition can no longer be expressed in terms of a "splitting completely" criterion, and the enumeration of primes for which it holds becomes worse. We are no longer able to deal with the case where the subgroup Γ of $\bar{E}(\mathbb{F}_p)$ has rank one, but can handle the case $r(\Gamma) \geqq 2$.

If q is a prime and $p \nmid \Delta q$, where Δ is the discriminant of E, then one can show that if Γ_p denotes the reduction of Γ mod p, $q|[\bar{E}(\mathbb{F}_p):\Gamma_p]$ if and only if $\sigma_q(p) \subset C_q$, where $\sigma_q(p)$ denotes the Frobenius symbol of p in the extension $L_q/\mathbb{Q}$ (a conjugacy class of G_q), and C_q is a certain subset of G_q which is closed under conjugation (see [3], p. 35). If p is a prime of good reduction which is inert in k, so that $|\bar{E}(\mathbb{F}_p)| = p + 1$, then $q|[\bar{E}(\mathbb{F}_p):\Gamma_p]$ if and only if $\sigma_q(p) \subset C_q'$, where $C_q' = \{\sigma \in C_q | \sigma|_k = -1\}$. For our purposes, the important fact about these objects is:

Lemma 4. *The size of G_q is $\asymp q^{2r(\Gamma)+2}$ and the sizes of C_q and C_q' are $\asymp q^{r(\Gamma)+1}$.*

Proof. This is proved in [3], p. 35 for G_q and C_q, and the same argument applies to C_q'.

For any n, let C_n (respectively C_n') be the set of elements of G_n whose restriction to L_q lies in C_q (respectively C_q') for all $q|n$; C_n and C_n' are also closed under conjugation. Then an inclusion-exclusion makes it reasonable to expect that the primes for which $\bar{E}(\mathbb{F}_p) = \Gamma_p$ have density

$$\delta(\Gamma) = \sum_{n \geqq 1} \mu(n) \frac{|C_n|}{|G_n|},$$

and indeed, such a result was proved in [3] under the assumption of the GRH, provided $r(\Gamma) \geqq 10$. Also, we expect $S'(\Gamma)$ to have density

$$\delta'(\Gamma) = \sum_{n \geqq 1} \mu(n) \frac{|C_n'|}{|G_n|}.$$

In order to enumerate the sets $S(\Gamma)$ and $S'(\Gamma)$, we will need the following estimates for the error term in the Chebotarev density theorem.

Proposition 2. *Let L/K be a Galois extension with Galois group G, and $C \subset G$ be closed under conjugation, and assume the GRH. Define*

$$\pi_C(x) = |\{\mathfrak{p} \text{ a prime of } K \text{ unramified in } L | \mathrm{Norm}_{K/\mathbb{Q}}\mathfrak{p} \leqq x \text{ and } \mathrm{Frob}_{L/K}(\mathfrak{p}) \subset C\}|.$$

Let d_K (resp. d_L) be the absolute value of the discriminant of K (resp. L), $n_K = [K:\mathbb{Q}]$, $n_L = [L:\mathbb{Q}]$, $n = [L:K]$, and $P(L/K)$ be the set of rational primes lying under the primes of K which ramify in L. Then

$$\pi_C(x) = \frac{|C|}{|G|} li\, x + O\left(\frac{|C|}{|G|} x^{1/2}(\log d_L + n_L \log x)\right),$$

and if Artin's holomorphy conjecture holds for L/K, then

$$\pi_C(x) = \frac{|C|}{|G|}\, li\, x + O\left(|C|^{1/2}x^{1/2}\log\left(nd_K^{1/n_K}\left(\prod_{p\in P(L/K)} p\right)x\right)\right).$$

Proof. The first assertion is due to Lagarias and Odlyzko [7], and the second is due to Murty, Murty, and Saradha [12].

A key ingredient in the proof of Theorems 2 and 3 is the following lemma.

Lemma 5. *Let*

$$\pi_n(x) = |\{p \leqq x | \sigma_p(p) \in C_n'\}|\,.$$

Then, under GRH, we have

$$\pi_n(x) = \frac{|C_n'|}{|G_n|}\, li\, x + O(|C_n'|^{1/2}x^{1/2}\log nx)\,.$$

Proof. Artin's holomorphy conjecture holds for the extension $L_n/\mathbb{Q}$ by Proposition 1. Therefore, by Proposition 2,

$$\pi_n(x) = \frac{|C_n'|}{|G_n|}\, li\, x + O\left(|C_n'|^{1/2}x^{1/2}\log\left([L_n:\mathbb{Q}]\left(\prod_{p\in P(L_n/\mathbb{Q})} p\right)x\right)\right),$$

and this gives the desired result since $[L_n:\mathbb{Q}] \ll n^{2r(\Gamma)+2}$ and only the primes of bad reduction and those dividing n can ramify in L_n.

Proof of Theorem 2. Let

$$N(x) = |\{p \leqq x | p \in S'(\Gamma)\}|;$$

we want to show $N(x) \geqq (1 - \log 2 - \varepsilon)\delta'(\Gamma)$. Following [3], we introduce the notation: $N(x, y) = |S(x, y)|$, where

$$S(x, y) = \{p \leqq x | p \text{ is inert in } k \text{ and } \sigma_q(p) \notin C_q \text{ for any } q \leqq y\}\,,$$

and $M(x, y, z) = |T(x, y, z)|$, where

$$T(x, y, z) = \{p \leqq x | p \text{ is inert in } k \text{ and } \sigma_q(p) \subset C_q \text{ for some } y < q < z\}\,.$$

Then, as in [3], we have

$$N(x, y) - M(x, y, 2x) \leqq N(x) \leqq N(x, y)$$

and by inclusion-exclusion,

$$N(x, y) = \sum \mu(m)\pi_m(x)\,,$$

where the summation is over all square-free integers all of whose prime factors are $\leqq y$. If

$$\delta'(y) = \sum \mu(m)\frac{|C_m'|}{|G_m|}\,,$$

with the range of summation as above, then using Lemma 5, we get

$$N(x, y) = \delta'(y)\frac{x}{\log x} + o(x/\log x)\,,$$

where y is sufficiently small and $y = y(x) \to \infty$ as $x \to \infty$ (e.g. $y = (1/6)\log x$), and $\delta(y) \to \delta'(\Gamma)$ as $y \to \infty$. If some prime in the range $x^{1/2}\log x < q < 2x$ divides $[\bar{E}(\mathbb{F}_p)\colon \Gamma_p]$, then $|\Gamma_p| < 2x^{1/2}\log x$, so by Lemma 14 of [3], we have

$$M(x, x^{1/2}\log x, 2x) = O(x/\log^2 x)\,.$$

Let $\tilde{\Gamma}$ be a rank 1 subgroup of Γ, so that

$$M(x, y, z) \leqq \tilde{M}(x, y, z)\,,$$

where $\tilde{M}(x, y, z)$ represents the same quantity as above but with Γ replaced by $\tilde{\Gamma}$. By Lemmas 4 and 5,

$$\tilde{M}(x, y, x^{1/4}/\log x) \leqq \sum_{y < q < x^{1/4}/\log x} ((1/q^2)\,li\,x + O(qx^{1/2}\log qx)) = o(x/\log x)$$

since $y \to \infty$ as $x \to \infty$, so we need only deal with $M(x, x^{1/4}/\log x, x^{1/2}\log x)$. In fact, it suffices to estimate

$$M'(x, x^{1/4}/\log x, x^{1/2}\log x) = |S(x, y) \cap T(x, x^{1/4}/\log x, x^{1/2}\log x)|\,.$$

It is convenient to introduce

$$M'(x, y; q) = |\{p \in S(x, y) | \sigma_q(p) \subset C_q'\}|\,.$$

If $\sigma_q(p) \subset C_q'$, then $q|[\bar{E}(\mathbb{F}_p)\colon \Gamma_p]$, so $p \equiv -1 \bmod q$. We can utilize this to obtain an upper bound for $M'(x, y; q)$. In the enumeration of $S(x, y)$ above, one has to replace the fields L_m, where all prime divisors of m are $\leqq y$, with the compositum of these fields with $\mathbb{Q}(\zeta_q)$, and the size of the conjugacy set *does not increase.* Moreover, for q sufficiently large, the field L_q, and hence $\mathbb{Q}(\zeta_q)$, is disjoint from all these L_m. This leads to

$$M'(x, y; q) \leqq \frac{\delta'(y)}{q-1}\,li\,x + O(x^{1/2}\log qx)\,,$$

for q sufficiently large, and using the fact that

$$\sum_{x^{1/4}/\log x < q < x^{1/2}/\log^2 x} \frac{1}{q-1} \sim \log\log(x^{1/2}/\log^2 x) - \log\log(x^{1/4}/\log x) \sim \log 2\,,$$

we obtain the estimate

$$M(x, x^{1/4}/\log x, x^{1/2}/\log^2 x) \leqq (1 - \log 2 - \varepsilon)\delta'(\Gamma)\,li\,x$$

for x sufficiently large. To handle the remaining interval $M(x, x^{1/2}/\log^2 x, x^{1/2}\log x)$, we use the Brun-Titchmarsh theorem (see [4]). Since if $q|[\bar{E}(\mathbb{F}_p)\colon \Gamma_p]$ then $p \equiv 1 \bmod q$, we have

$$M(x, x^{1/2}/\log^2 x, x^{1/2}\log x) \ll \sum_{x^{1/2}/\log^2 x < q < x^{1/2}\log x} \frac{x}{q\log(x/q)}\,,$$

which is easily seen to be $o(x/\log x)$. This completes the proof of Theorem 2.

Proof of Theorem 3. The proof is very similar to that given above, so we give only a sketch. One does an initial sieve as before and obtains a bound

$$M(x, y, x^{1/4}/\log x) = o(x/\log x)\,,$$

and since $r(\Gamma) \geqq 6$, Lemma 14 of [3] yields

$$M(x, x^{1/4} \log x) = o(x/\log x) .$$

The remaining interval is handled as above using the Brun-Titchmarsh theorem for inert primes and a generalization of it to number fields for primes which split in k, as in [3].

5. Generation of $\bar{E}(\mathbb{F}_p)$ in the non-CM case

We now suppose that E does not have CM, and ask how often $\bar{E}(\mathbb{F}_p) = \Gamma_p$. If $p \nmid q\Delta$, then we have as before that $q|[\bar{E}(\mathbb{F}_p): \Gamma_p]$ if and only if $\sigma_q(p) \subset C_q$, where C_q is a conjugacy subset of $G_q = \mathrm{Gal}(L_q/\mathbb{Q})$, and $L_q = \mathbb{Q}(E[q], \frac{1}{q}\Gamma)$, but in this case, the sizes of G_q and C_q are bigger; we have:

Lemma 6. *We have* $|G_q| \asymp q^{2r(\Gamma)+4}$ *and* $|C_q| \asymp q^{r(\Gamma)+3}$.

Proof. See [3], p. 35. The difference from Lemma 4 arises from the fact that the Galois group of $\mathbb{Q}(E[q])/\mathbb{Q}$ is almost always $GL_2(\mathbb{Z}/q\mathbb{Z})$.

We expect $S(\Gamma)$ to have density $\delta(\Gamma)$, where $\delta(\Gamma)$ is as in section 4. Because of the large size of the conjugacy sets involved and the resulting worsening of the error terms in the Chebotarev density theorem, one can only prove this in case $r(\Gamma) \geqq 18$, under the GRH (see [3]). Using the improved error term of [12], one can improve this to $r(\Gamma) \geqq 11$, under the assumption of Artin's holomorphy conjecture.

We are able to bring the rank down even further by not insisting on producing a set of density $\delta(\Gamma)$ but rather a positive multiple of it, as we did in section 3.

Proof of Theorem 4. We will use the same notation as in the proof of Theorem 2. For $r(\Gamma) \geqq 7$, we are able to handle $M(x, x^{2/9+\varepsilon}, 2x)$ using Lemma 14 of [3], and after doing an initial sieve as before, are left with having to estimate $M(x, y, x^{2/9+\varepsilon})$, where $y \to \infty$ as $x \to \infty$. We instead estimate $\tilde{M}(x, y, x^{1/4+\varepsilon})$, corresponding to a rank-one subgroup $\tilde{\Gamma}$ of Γ. Using Proposition 2, one can show under the GRH that

$$\tilde{M}(x, y, x^{1/10-\varepsilon}) = o(x/\log x) ,$$

since the size of the conjugacy class involved is $\asymp q^4$ by Lemma 6, and hence the error terms are essentially $O(q^4 x^{1/2})$. One must use the fact, which is proved as in [3, Lemma 7], that $d_{\tilde{L}_n} \ll [\tilde{L}_n: \mathbb{Q}] \log n$. Under the additional assumption of Artin's holomorphy conjecture, one can show using Proposition 2 that

$$\tilde{M}(x, y, x^{1/6-\varepsilon}) = o(x/\log x) .$$

To estimate the remaining portion of the tail, we impose the condition $q|[\bar{E}(\mathbb{F}_p): \tilde{\Gamma}_p]$ into the initial sieve. Letting m be a square-free integer all of whose prime factors are $\leqq y$, we proceed to estimate the number of primes p for which $\sigma_m(p) \subset C_m$ and $q|[\bar{E}(\mathbb{F}_p): \tilde{\Gamma}_p]$. Let F_q be the (non-Galois) extension of $\mathbb{Q}$ of degree q^2 obtained by adjoining to $\mathbb{Q}$ some value of $\frac{1}{q}X$, where X is a generator of $\tilde{\Gamma}$. From [3], p has q first degree primes above it in F_q or splits completely in $\mathbb{Q}(E[q])$. The latter condition has the effect of introducing essentially q^4 (the degree of $\mathbb{Q}(E[q])$

over $\mathbb{Q}$) into the denominator of the main term and only a factor of $\log q$ into the error term. Also, applying Proposition 2 to the Galois extension $F_q L_m / F_q$, with the conjugacy set C_m^* which is the extension of C_m, we get the following estimate for the number of first-degree primes of F_q with norm $\leqq x$ and whose Frobenius symbol lies in C_m^*:

$$\frac{|C_m|}{|G_m|} li\, x + O(|C_m| q^2 x^{1/2} \log mqx) .$$

Here we have used the estimates

$$d_{L_m} \ll [L_m : \mathbb{Q}] \log m, \; d_{F_q} \ll [F_q : \mathbb{Q}] \log q ,$$

which can be easily proved as in [3] because of the limited ramification in these fields. Using the same notation as before, we conclude that

$$M(x, x^a, x^{2/9+\varepsilon}) \leqq \delta(\Gamma) \sum_{x^a < q < x^{2/9+\varepsilon}} \left(\frac{1}{q} + \frac{1}{q^4}\right) li\, x + O(|C_m| q x^{1/2} \log mqx) .$$

Choosing y sufficiently small but still tending to ∞ (e.g. $y = c \log\log x$), and using

$$\sum_{x^a < q < x^{2/9+\varepsilon}} \frac{1}{q} \sim \log \frac{2/9 + \varepsilon}{\alpha} ,$$

with $\alpha = 1/10 - \varepsilon$ or $1/6 - \varepsilon$, we obtain the desired result.

6. Concluding remarks

If E is an elliptic curve over a number field K, then one can still ask the question of how often $E(F_v)$ is cyclic where v is a prime ideal of K. The method of proof of Theorem 1 requires a corresponding sieve result. More precisely, we need to know that primes that split completely in K are well-distributed in arithmetic progressions. If K is an abelian extension of $\mathbb{Q}$, this amounts to putting additional congruence conditions on the primes so that the theorem of Fouvry and Iwaniec of sieve theory can be employed without alteration. Hence, if K is abelian over $\mathbb{Q}$, then there are at least $\gg x/\log^2 x$ prime ideals v of K such that $\mathrm{Norm}(v) \leqq x$ and $E(F_v)$ is cyclic, provided $E[q] \not\subset E(K)$ for any prime q. However, if K is not abelian over $\mathbb{Q}$, one would have to introduce non-abelian conditions into the sieve theorem of Fouvry and Iwaniec. This has been done for the classical theorem of Bombieri and Vinogradov (see [11]). But this would only produce primes p of the desired type such that all the odd prime factors of $p - 1$ are $\gg p^{1/4-\varepsilon}$. This is not sufficient for our needs since the fact that the exponent is $> 1/4$ was crucial to our argument.

References

1. Fouvry, E., Iwaniec, H.: Primes in arithmetic progression. Acta Arith. **42**, 197–218 (1983)
2. Gupta, R., Murty, M.R.: A remark on Artin's conjecture. Invent. Math. **78**, 127–130 (1984)
3. Gupta, R., Murty, M.R.: Primitive points on elliptic curves. Compos. Math. **58**, 13–44 (1986)

4. Halberstam, H., Richert, H.E.: Sieve Methods. London Mathematical Society Monographs No. 4. New York: Academic Press 1975
5. Heath-Brown, D.R.: Artin's conjecture for primitive roots. Q. J. Math., Oxf. II. Ser. **37**, 27–38 (1986)
6. Hooley, C.: On Artin's conjecture. J. Reine Angew. Math. **225**, 209–220 (1967)
7. Lagarias, J., Odlyzko, A.: Effective versions of the Chebotarev density theorem. In: Algebraic Number Fields Fröhlich, A. (ed.). New York: Academic Press 1977, pp. 409–464
8. Lang, S., Trotter, H.: Primitive points on elliptic curves. Bull. Am. Math. Soc. **83**, 289–292 (1977)
9. Murty, M.R.: On Artin's conjecture. J. Number Theory **16**, 147–168 (1983)
10. Murty, M.R.: On the supersingular reduction of elliptic curves. Proc. Indian Acad. Sci., Math. Sci. **97**, 247–250 (1987)
11. Murty, M.R., Murty, V.K.: A variant of the Bombieri-Vinogradov theorem. In: Conference Proceedings of the CMS Vol. 7 (1987), AMS publication
12. Murty, M.R., Murty, V.K., Saradha, N.: Modular forms and the Chebotarev density theorem. Am. J. Math. **110**, 253–281 (1988)
13. Serre, J.-P.: Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. Invent. Math. **15**, 259–331 (1972)
14. Serre, J.-P.: Resumé de cours (1977). See: Oeuvres. Berlin-Heidelberg-New York: Springer 1986
15. Serre, J.-P.: Linear Representations of Finite Groups. Berlin-Heidelberg-New York: Springer 1977
16. Silverman, J.: The Arithmetic of Elliptic Curves. Berlin-Heidelberg-New York: Springer 1986

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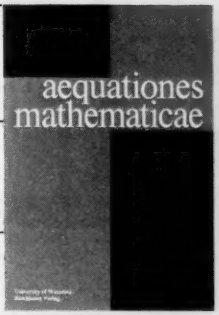

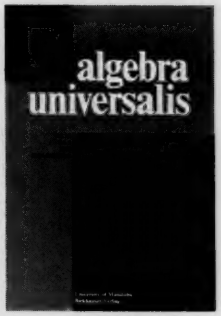

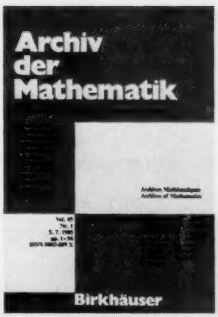

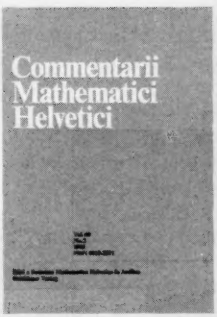

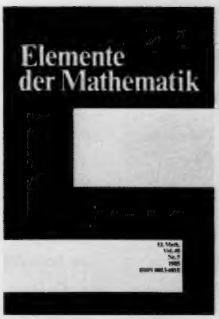

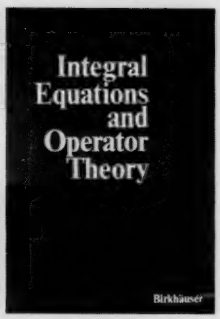

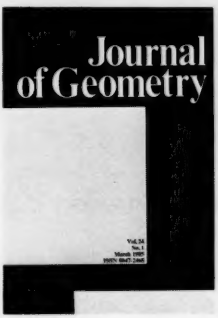

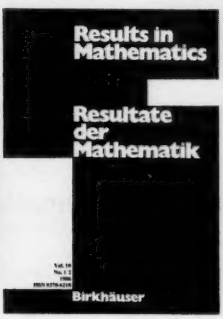

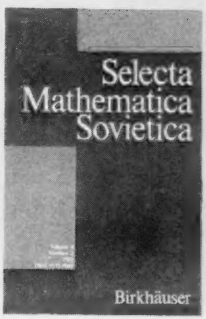

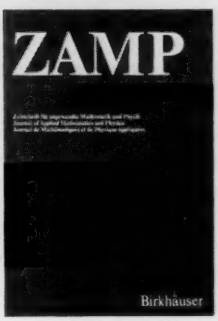

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